# Dobrushin Interfaces via Reflection Positivity

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**Abstract**

We study the interfaces separating different phases of 3D systems by means of the Reflection Positivity method. We treat discrete non-linear sigma-models, which exhibit power-law decay of correlations at low temperatures, and we prove the rigidity property of the interface.

Our method is applicable to the Ising and Potts models, where it simplifies the derivation of some known results. The method also works for large-entropy systems of continuous spins.

# 1 Introduction

The first example of a pure state describing the coexistence of phases separated by an interface was discovered by R. Dobrushin in 1972, [D72]. There he was studying the low temperature 3D Ising model. He was considering the Ising spins in a cubic box $V_N$ with $(\pm)$-*boundary condition* $\sigma^{\pm}$: all spins

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of $\sigma^{\pm}$ are $(+)$ in the upper half-space and $(-)$ in the lower half-space. Such a boundary condition forces an interface $\Gamma$ into $V_N$, separating the $(+)$-phase from the $(-)$-phase. Dobrushin has shown that in the thermodynamic limit $N \to \infty$ the distribution of $\Gamma$ goes to a proper limit (in contrast with the 2D case). This limit describes the behavior of the surface separating the $(+)$- and the $(-)$-phases. His method of analysis was what is now called the cluster expansion, based on Pirogov-Sinai Contour Functional theory. Later on, this approach was applied to other discrete models in [HKZ, CK, GG].

The question of coexistence of phases for systems with continuous symmetry was addressed in [FP]. It was found there that the analogous states for the $XY$-model do not exist, and that the surface tension between two magnetized phases vanishes. Other systems were not studied in the literature. There are probably two reasons for that:

1. most systems with continuous symmetry do not display the above Ising-type rigid interface separating different phases,
2. the Pirogov-Sinai theory (PS) "does not work" for continuous symmetry systems, while the (only) alternative method – the Reflection Positivity (RP) – works just for periodic boundary conditions, and therefore one can not handle boundary conditions of the type $\sigma^{\pm}$ needed in order to create the interface.

In order to illustrate the first point, let us consider the low-temperature 3D classical $XY$ model, defined by the Hamiltonian

$$H(\varphi) = -\sum_{\substack{x,y\in\mathbb{Z}^3, \\ |x-y|=1}} \cos(\varphi_x - \varphi_y), \tag{1}$$

where the spins $\varphi_{\cdot}$ are taking values on the circle $\mathbb{S}^1 = \mathbb{R}^1 \mathrm{mod}\,(2\pi)$. As was established in the seminal paper [FSS], this model has a continuum of low-temperature magnetized phases, $\langle\cdot\rangle_\alpha$, $\alpha \in \mathbb{S}^1$. One can try to create a state of coexistence of two phases by using the $(\pm)$-boundary condition $\varphi^{\pm}$, which assigns the value 0 to spins in the upper half-space and the value $\pi$ in the lower half-space. However, as the comparison with the Gaussian case shows, one expects the thermodynamic limit of that state to be the mixture state, $\frac{1}{2}\left(\langle\cdot\rangle_{\pi/2} + \langle\cdot\rangle_{3\pi/2}\right)$, with no interface emerging. The $XYZ$ model is defined by the same Hamiltonian $(1)$, but the variables $\varphi$-s are

taking values on the sphere $\mathbb{S}^2 \in \mathbb{R}^3$, and the difference $\varphi_x - \varphi_y$ is just the angle between $\varphi_x$ and $\varphi_y$. Let $(\psi, \theta)$ be the "Euler angles" coordinates on $\mathbb{S}^2$, $\psi \in \mathbb{S}^1$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Again, at low temperatures there are extremal Gibbs states $\langle \cdot \rangle_{(\psi,\theta)}$, $(\psi, \theta) \in \mathbb{S}^2$. The $(\pm)$-boundary condition $\varphi^{\pm}$ is now the configuration assigning the value $\theta = \frac{\pi}{2}$ (north pole) to the upper half-space, and $\theta = -\frac{\pi}{2}$ (south pole) to the lower half-space. We expect that the corresponding finite-volume state $\langle \cdot \rangle_{\varphi^{\pm}}^{N}$ converges weakly, as $N \to \infty$, to the mixture $\int \langle \cdot \rangle_{(\psi,\theta=0)}\, d\psi$.

Still, we believe that Dobrushin states for some systems with continuous symmetry do exist. One likely example is the so-called non-linear sigma-model, considered recently in [ES1, ES2]. Its Hamiltonian is given by

$$H(\varphi) = -\sum_{\substack{x,y \in \mathbb{Z}^d, \\ |x-y|=1}} \left( \frac{1 + \cos(\varphi_x - \varphi_y)}{2} \right)^p, \tag{2}$$

with $\varphi_x \in \mathbb{S}^1$. For $p$ large enough – i.e. when the potential well is narrow enough – this model exhibits the following behavior: at high temperatures it has unique Gibbs state (the chaotic state). At low temperatures in 2D it presumably has the Kosterlitz-Thouless phase with power-law correlation decay, which can be obtained by the methods of the paper [FS]. At low temperatures in 3D it should have infinitely many ordered Gibbs states, indexed by magnetization, as the results of [FSS] suggest. Moreover – and that is the main result of [ES1] – there exists a critical temperature $T_c = T_c(p, d)$, at which we have the coexistence of the chaotic state and the ordered state(s). (Of course, all these states are translation-invariant.) The results of [ES1] are valid in any dimension $d \geq 2$. We believe that in dimension $d = 3$ at the critical temperature $T_c$ the system possesses also non-translation-invariant states, describing the coexistence of ordered states and chaotic state, with the rigid interface separating them.

The present paper was started as an attempt to prove the above conjecture. Unfortunately, we are currently unable to complete this program. (Our partial results in this direction are briefly described at the end of this introduction.) However, we are able to study the interfaces in some discrete approximations of the non-linear sigma-model and other models of this type. By discrete approximation we mean here the following. Let $H(\varphi) = -\sum_{\substack{x,y \in \mathbb{Z}^3, \\ |x-y|=1}} U(|\varphi_x - \varphi_y|)$ be the Hamiltonian for the continuous spin model, $\varphi_x \in \mathbb{S}^1$, with free measure $d\varphi$. Then its discrete approximation is

given by the Hamiltonian

$$H(\sigma) = -\sum_{\substack{x,y \in \mathbb{Z}^3, \\ |x-y|=1}} U(|\sigma_x - \sigma_y|), \tag{3}$$

with $\sigma_x \in \mathbb{Z}_q \subset \mathbb{S}^1$, where the group $\mathbb{Z}_q$ is equipped with counting measure. The integer $q$ is the parameter of the approximation. (One can call the resulting model as *the clock-model,* corresponding to the interaction $U$.)

If the function $U$ has unique nondegenerate minimum on $\mathbb{S}^1$, then the resulting $\mathbb{Z}_q$-model at low temperatures is Potts-like, and thus has properties quite different from the continuous symmetry system. The situation becomes much more interesting if the *minimum of $U$ is degenerate* and, moreover, the *minimal value is attained along a (small) segment*, while the discretization parameter $q$ is large. Then the properties of such a system are quite similar to the one with continuous symmetry. Unlike the Potts model, the ground states of our Hamiltonian (3) are infinitely degenerate. We believe that in the 3D case at low temperatures (as well as at zero temperature) such a model exhibits spontaneous magnetization, while the truncated correlations decay as a power law. Hopefully one can establish this conjectured behavior by a suitable version of the infrared bounds. In the 2D case we believe that "Mermin-Wagner" theorem holds, so that the magnetization is zero, even at zero temperature. We expect the correlation decay to be a power law. Our 2D conjecture at zero temperature is close in spirit to the results of R. Kenyon [K] on 2D tilings, while for positive low temperatures its behavior looks to us to be similar to that of the intermediate phase of the classical clock-model, established in [FS]. Another model with similar features was considered by M. Aizenman, [A].

The methods of the cited papers [ES1, ES2] can be easily adapted to prove that in dimension $d \geq 2$ the structure of the phase diagram for the Hamiltonian (3), with the function $U$ having deep and $\varepsilon$-narrow well (possibly with a flat bottom) and $q$ large enough, has the same features as for the "very" non-linear sigma-model: at high temperatures it has unique Gibbs state, while at low temperatures it has (one or more) Gibbs states "with local order", which means that the probability of seeing the discrepancy: $|\sigma_i - \sigma_j| \geq 3\varepsilon$ at two n.n. sites is small. Moreover, there exists a temperature $T_c(q)$ at which the high-T chaotic state coexists with the low-T locally-ordered state(s).

The main result of the present paper is the rigidity property of the chaos/order interface once the dimension $d$ is at least 3. Namely, we show

that if the two phases are put into coexistence at the transition temperature $T_c(q)$ by applying suitable boundary conditions in a given volume, then the interface between them is rigid, and its height function exhibits the long-range order. Since the proof of this result is quite involved, we will establish it in the present paper only for the simplest model of the above type, defined below, (6).

We will now comment on the method we use to study our problem. Presently there are two techniques to study phase transitions: the Pirogov-Sinai theory and the method of Reflection Positivity. It seems unlikely that our model can be treated by PS-theory, since we have here infinite degeneracy of the ground states and we expect power-law decay of correlations. On the other hand, the applications of the RP method rely on the study of the states with periodic boundary conditions. In the phase coexistence regime such a state is not ergodic, and its ergodic decomposition allows one to study various pure states – but only *some states,*since the non-translation-invariant states do not contribute to the state with periodic boundary conditions.

Notwithstanding the above discussion, our method of proof will be that of Reflection Positivity. But in order to study the chaos/order interface, we will use RP not with periodic boundary conditions, but with mixed ones; namely, we impose periodic boundary conditions only in two (horizontal) dimensions, and we leave the third (vertical) dimension "free" to impose fixed spins boundary conditions in the third dimension. In other words, we consider the cylindric boundary conditions topology, $\mathbb{T}_N^2 \times [0, L]$, and we impose ordered boundary conditions on the top of the cylinder and chaotic boundary conditions on its bottom. Of course, the resulting state will be RP only with respect to reflections in vertical planes, but that will be sufficient for our purposes. This restricted Reflection Positivity is the main technical innovation of this paper.

**Our main result** will be that the so constructed state at $T = T_c$ necessarily possesses an interface, separating the ordered and the disordered phases, which interface is rigid in the sense of [D72]: it has a well-defined (random) global height, while the deviations from it happen at any given location with a small probability.

One usual advantage of RP method and the chess-board estimates is that their technical implementations are usually quite simple, as compared with the Pirogov-Sinai theory. In this respect we have to note that the *restricted* RP is already more involved technically and requires a detailed study of various spatially extended defects, not present in the usual applications of

RP.

Our technique enables one to study also the continuous symmetry case, and to obtain similar results in a 3D slab $\mathbb{Z}^2 \times [0, L]$ : with order-disorder boundary conditions and for a suitable narrow-well interaction $U$ one has the chaos/order rigid interface at the critical temperature $T_c$. However, the technical limitations of our approach are such that the width $\varepsilon$ of the potential well depends on the width $L$ of the slab, with $\varepsilon (L) \to 0$ as $L \to \infty$. Therefore, in contrast to the discrete symmetry case, we can not take the full thermodynamic limit $N \to \infty$, $L \to \infty$. The details will be published separately, see [V1].

**Other models.** Finally we remark that our technique, applied to the 3D Ising or Potts models, allows one to obtain simpler proofs of the rigidity of their interfaces. Indeed, since in these models the ground states are non-degenerate, our machinery simplifies a lot, and the resulting proofs are relatively short. We can also treat various 3D real valued random fields. For example, we can study the double-well case, defined by the Hamiltonian

$$H(\psi) = \sum_s \left(\psi_s^2 - 1\right)^2 + \sum_{s,t \text{ n.n.}} \left(\psi_s - \psi_t\right)^2 . \tag{4}$$

We can show that at low temperatures this system possesses rigid interface separating the plus-phase, where $\psi \approx +1$, from the minus-phase, where $\psi \approx -1$. Another case of interest is the model with extra local minimum of the energy, considered in [DS], where

$$H(\psi) = \sum_s \Phi(\psi_s) + \sum_{s,t \text{ n.n.}} \left(\psi_s - \psi_t\right)^2 . \tag{5}$$

Here the potential $\Phi$ has a (unique) global minimum, which is narrow, and an additional local one, which has to be relatively wide. Then, as it is shown in [DS], such a model undergoes a phase transition at some temperature $T_{cr}(\Phi)$, at which temperature one has a coexistence of the low-energy phase, corresponding to the global minimum, with high entropy phase, corresponding to the local minimum. In dimension 3 we can show that at this temperature this model exhibits rigid interface separating the above two phases.

We want to stress that the above stated results for the models (4) and (5) are technically simpler than the corresponding statements for the system (3) and its discrete versions. Indeed, while in the models (4) and (5) one has exponential decay of correlation due to the positive mass of the potential

wells, in (3) and its discrete version we expect power law decay. This is why in the present paper we concentrate on the last model. The corresponding results for (4) and (5) will be published separately, [SV].

The organization of the paper is the following:

The next section contains the definition of the model we study and the formulation of the main result. Section 3 contains the main steps of the proof. We introduce there the gas of defects of the interface, and we use Reflection Positivity and the chess-board estimates to reduce the study of the local defects to the study of defect sheets. Some defects do not contribute to the weight of the interface, so to control these we have to glue them in pairs by means of the gluing transformation. The Sections 4 and 5 contain the needed combinatorial-energy properties of various defect sheets. The last Section 6 contains the final steps of the proof of our main result.

# 2 The Main Result

In what follows we will consider the 3D lattice model with spins $\sigma_i$ taking values in the additive group $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. We will equip $\mathbb{Z}_q$ with the counting measure. Let $\sigma = \{\sigma_i : \sigma_i \in \mathbb{Z}_q, i \in \mathbb{Z}^3\}$ be a configuration of our model. The Hamiltonian of our system is given by

$$H(\sigma) = -\sum_{i \sim j} \mathbb{I}_{|\sigma_i - \sigma_j| \leq 1}, \tag{6}$$

where the summation goes over nearest neighbors. Clearly, the interaction and the Hamiltonian are $\mathbb{Z}_q$-invariant. (In terms of Section 1, the Hamiltonian (6) corresponds to the model (3) with interaction having a well of width $\varepsilon = \frac{3}{q}$.)

Let us define the notion of order:

**Definition 1 (Ordered bonds)** *A bond $i \sim j$ of our lattice $\Lambda_{N,L}$ is called ordered in $\sigma$ iff $|\sigma_i - \sigma_j| \leq 1$. Otherwise it is called disordered.*

Using a technique similar to [ES1, ES2], one can show that for $q$ large enough the above model undergoes a first-order phase transition in temperature. Namely, the following theorem holds:

**Theorem 2** *There exists a temperature $T_c = T_c(q)$, at which the Hamiltonian (6) has at least two Gibbs states: the ordered state $\langle\cdot\rangle^o_{T_c}$ and the disordered state $\langle\cdot\rangle^d_{T_c}$. They are characterized by the properties:*

$$\left\langle \mathbb{I}_{|\sigma_i-\sigma_j|\leq 1}\right\rangle^d_{T_c} \leq p(q), \tag{7}$$

$$\left\langle \mathbb{I}_{|\sigma_i-\sigma_j|\leq 1}\right\rangle^o_{T_c} \geq 1-p(q), \tag{8}$$

*where $i, j$ is any bond of $\mathbb{Z}^3$, while $p(q)$ goes to zero as $q \to \infty$. (Incidentally, the critical temperature $T_c(q)$ goes to zero as $q \to \infty$.)*

NOTE. We believe that in 3D the state $\langle\cdot\rangle^o_{T_c}$ is not pure, and is a mixture of $q$ states with different values of magnetization.

The purpose of our work is the study of the interface between the ordered and disordered phases of the Hamiltonian (6) at the critical temperature $T_c(q)$, put into coexistence by suitable boundary conditions. The construction of the corresponding non-translation-invariant states will be discussed in another publication, [V].

To study the interfaces we will consider special boxes and we will impose special boundary conditions, which will force the interface into the box. Namely, we will take the boxes $\Lambda_{N,L} \subset \mathbb{Z}^3$ :

$$\Lambda_{N,L} = \{(x, y, z); 0 \leq x, y \leq N;\ 0 \leq z \leq L+1\},$$

and we will impose the *periodic boundary conditions in $x$ and $y$ directions*. In other words, we think about the box $\Lambda_{N,L}$ as a product of the torus $\mathbb{T}_N$ and a segment. In what follows we suppose that $N$ is even. The boundary of $\Lambda_{N,L}$ has two components, and we denote them by

$$\mathcal{P}^o = \Lambda_{N,L} \cap \{z = L+1\} \text{ and } \mathcal{P}^d = \Lambda_{N,L} \cap \{z = 0\}.$$

We will impose boundary conditions on $\mathcal{P}^o$ and $\mathcal{P}^d$, which (hopefully) would bring the order-disorder interface into $\Lambda_{N,L}$. So we fix a value $s \in \mathbb{Z}_q$, and we impose on $\mathcal{P}^o$ the *ordered* boundary condition $\sigma_{\text{ord}} = \{\sigma_{i,j,L+1} \equiv s\}$. We also fix four values: $s_{00} = 0$, $s_{01} = [q/4]$, $s_{10} = [3q/4]$ and $s_{11} = [q/2]$ in $\mathbb{Z}_q$, and we impose the *strongly disordered* boundary condition $\sigma_{\text{disord}} = \{\sigma_{a+2i,\,b+2j,\,0} = s_{ab}, a, b = 0, 1\}$ on $\mathcal{P}^d$. The resulting boundary condition will be called the *order-disorder b.c.*

In what follows we will be interested in the Gibbs states in $\Lambda_{N,L}$, corresponding to the Hamiltonian (6), with these order-disorder b.c. at inverse

temperature $\beta$. They will be denoted by $\mu_{N,L}^{\beta,q}$, while by $Z_{N,L}^{\beta,q}$ we denote the corresponding partition function.

To formulate our results we need some more definitions. Let a configuration $\sigma$ in $\Lambda_{N,L}$ be fixed.

**Definition 3 (Pure and frustrated cubes)** *We will call an elementary cube of our lattice $\Lambda_{N,L}$ frustrated in $\sigma$, if it has both ordered and disordered bonds among its (twelve) bonds. Otherwise it will be called pure. Any pure cube is either ordered or chaotic, in obvious sense.*

The set of all frustrated cubes of $\sigma$ will be denoted by $\mathcal{F}(\sigma)$.

**Definition 4 (Contours, 3D-interfaces)** *A connected component of $\mathcal{F}(\sigma)$ is called a 3D* interface*, iff it separates $\mathcal{P}^o$ and $\mathcal{P}^d$. Otherwise it is called a* contour*.*

**Remark 5** *Here two cubes are called connected, if they share at least one bond.*

The union of all the 3D interfaces of $\sigma$ will be denoted by $I(\sigma)$. The complement $\Lambda_{N,L} \setminus I(\sigma)$ has several connected components; each one of them is occupied by a phase – ordered or chaotic. The type of the phase in any of these components is defined by the type of the elementary cube on its inner boundary; inside the components the phases might have of course frustrated contours.

We need the following topological fact:

**Proposition 6 (Existence of a 3D-interface)** *With the order-disorder b.c., defined above, each configuration has at least one 3D-interface.*

This obvious claim in fact requires a proof, as was pointed out by G. Grimmett, [G]. One is given in [GG], though it also can be deduced from known results of homotopy theory, see, e.g. [D].

Now we will define the boundary surface, which rigidity we will prove below:

**Definition 7 (2D-interface)** *Let $\sigma$ be a configuration in $\Lambda_{N,L}$, with order-disorder b.c. imposed. The complement $\Lambda_{N,L} \setminus I(\sigma)$ has several (at least two – containing $\mathcal{P}^d$ and $\mathcal{P}^o$) connected components. Let us consider all its*

*disordered components. (There is at least one such component – the one containing the boundary $\mathcal{P}^d \subset \Lambda_{N,L}$.) We denote their union by $\mathcal{D}(\sigma)$; this is the* disordered phase *region. Denote by $\partial\mathcal{D}$ all the plaquettes which belong both to elementary cubes in $\mathcal{D}$ and to elementary cubes in $I(\sigma)$. It can have several connected components. Let $B(\sigma)$ be the union of these components, each of which separates $\mathcal{P}^d$ and $\mathcal{P}^o$. It will be called the* 2D-interface, *or just* the interface.

*A collection of plaquettes $B$ will be called* admissible *if $B = B(\sigma)$ for some configuration $\sigma$.*

Let us denote by $\Pi : B(\sigma) \to \mathcal{P}$ the orthogonal projection onto the plane $\mathcal{P} = \{z = 0\}$. A point $\bar{M}$ of the surface $B(\sigma)$ will be called *regular*, if the preimage of its projection $\Pi^{-1}\left(\Pi\left(\bar{M}\right)\right)$ consists of exactly one point, which is $\bar{M}$ itself. The plaquette $p$ of $B$, containing $\bar{M}$ will be then also called regular, as well as the point $M = \Pi\left(\bar{M}\right) \in \mathcal{P}$ and its plaquette. A *ceiling* is a maximal connected component of regular plaquettes. We split the complement of ceilings of $B$ into connected components, which will be called *walls*. Note that all plaquettes of a ceiling $\mathcal{C}$ necessarily belong to the same horizontal plane $\{z = h(\mathcal{C})\}$, so the *height of a ceiling* $h(\mathcal{C})$ is well defined. The height $h(M)$ of the regular point $M \in \mathcal{P}$ is defined in the obvious way. If the point $M \in \mathcal{P}$ is not regular, we put $h(M) = \infty$ by definition. The regular points $M$ of the plane $\mathcal{P}$ also can be splitted into connected components. Let $R(\sigma) \subset \mathcal{P}$ be the one with the largest area. (If there are several such, we choose one of them.) The set $R(\sigma)$ will be called the *rigidity* set of $\sigma$. The preimage $\bar{\mathcal{C}}(\sigma) = \Pi^{-1}(R(\sigma))$ is (contained in one of) the largest ceiling of $B(\sigma)$.

Our main result states that, typically, the rigidity set is *very* big:

**Theorem 8**

- ***Rigidity.*** *Let $q > q_0$, with $q_0$ being large enough. Let our box $\Lambda_{N,L}$ has even width $N$, while the height $L$ does not exceed $\exp\left\{N^{2/3}\right\}$. Then for every $\beta$*
$$\mu_{N,L}^{\beta,q}\left\{\frac{|R(\sigma)|}{N^2} > 1 - a(q)\right\} \to 1$$
*as $N \to \infty$, for some $a(q) > 0$, with $a(q) \to 0$ as $q \to \infty$. In particular, the surface $B$ has typically only one connected component.*

- ***Long-range order.*** *The function $h(M)$ is the long-range order parameter: if $M', M''$ are two arbitrary points in $\mathcal{P}$, then the probability of the event*

$$\mu_{N,L}^{\beta;q}\left\{h\left(M'\right)=h\left(M''\right) \text{ and are finite}\right\} \rightarrow 1$$

  *as $q \rightarrow \infty$, uniformly in $N$ and $M', M''$ and for every $\beta$.*

Of course, for most values of the temperature this result is not very surprising. Indeed, if $T > T_{cr}$, say, then the box $\Lambda_{N,L}$ will be filled with disordered phase, while the surface $B(\sigma)$ is pressed to stay in the vicinity of the $P^o$-component of the boundary. Our result is of real interest precisely at criticality, since at $T = T_{cr}$ the surface $B(\sigma)$ stays away from the boundaries of the box $\Lambda_{N,L}$ due to the entropic repulsion. We expect that at criticality the location $h\left(\bar{\mathcal{C}}\left(\sigma\right)\right)$ of the interface $B(\sigma)$ is distributed approximately uniformly in the segment $\left[C \ln N, L - C \ln N\right]$. The details will be given in [V].

We would like to comment that the power of the RP method lies in the property that one can make statements about the behavior at the critical point by establishing some features for general temperatures. Indeed, it would be very difficult for us to work precisely at the critical temperature, since we do not even know its exact value.

The main step towards the proof of rigidity is the control of the fluctuations of the interface with respect to the optimal flat shape. We thus make the following definition:

**Definition 9** *Let $B$ be the interface, and $D \subset B$ be any collection of plaquettes. We define the* weight *of $D$ to be $w(D) = |D| - |\Pi(D)|$, where $|\cdot|$ is the number of plaquettes in the collection.*

We have the following estimate:

**Theorem 10 (Peierls estimate)** *Suppose that $N$ is even. Then, for all $\beta, L$ and all collections of plaquettes $D$,*

$$\mu_{N,L}^{\beta,q}\left(B : D \subset B\right) \leq a^{w(D)},$$

*where $a = a(q)$ goes to $0$ when $q \rightarrow \infty$.*

# 3 Proof of the Theorem 10

## 3.1 Settings for reflection positivity, construction of the blobs

In order to set the framework for reflection positivity, we consider the system as a spin-system on the 2-dimensional torus $\mathbb{T}_N$, where at each site of $\mathbb{T}_N$ we have a random variable taking values in $(\mathbb{Z}_q)^L$ (we recall that $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$).

It is straightforward to see that $\mu_{N,L}^{\beta;q}$ is reflection positive with respect to the group generated by the reflections in the lines passing through the sites of the torus, see any of the RP papers [FL, FILS], or the review [S].

Let $p \subset \mathbb{T}_N$ be any plaquette. Its full preimage $c = \Pi^{-1}(p) \subset \Lambda_{N,L}$ will be called a column. The set of all columns will be denoted by $C_N$. Any *horizontal* plaquette $\Lambda_{N,L}$ belongs to a well defined column, but for (some) *vertical* plaquettes we will make a $\sigma$-dependent choice. We assume the following convention: let $P$ be a vertical plaquette, separating a frustrated cube of configuration $\sigma$ from a pure disordered one; then we say that $P$ belongs to the column containing the frustrated cube but not to the column containing the pure one. Now for any column $c \in C_N$, we define $B_c = B_c(\sigma)$ to be the set of plaquettes of $B(\sigma)$ contained in $c$.

**Definition 11** *We define the blobs of $\sigma$ in $c$ to be the connected components of $B_c$. We will denote by $\mathfrak{B}(B_c) = (b_1, \dots, b_r)$ the set of blobs in the column $c$ for the collection $B(\sigma)$, enumerated upwards.*

## 3.2 Application of the chessboard estimate

In the first three subsections of this section we will reduce the Peierls estimate – the estimate of a local event, see (9), – to an estimate of a global event $(\hat{\pi}_\tau)_c^N$, see (14). The remaining two subsections describe the splitting of $(\hat{\pi}_\tau)_c^N$ into defects and their pairing.

Let $\sigma$ be some configuration. We distinguish several kinds of blobs in $\mathfrak{B}(B_c) = (b_1, \dots, b_r)$, as we move upwards. The blob $b_i$ has:

- type $h-$ ($h+$), if $b_i$ begins (ends), as one ascends, with a horizontal plaquette, the rest being vertical; if $b_i$ consists of just a single plaquette, then it is of type $h-$ ($h+$) if the cube below (above) it is pure disordered;

- type $h-+$, if $b_i$ begins *and* ends with a horizontal plaquette, the rest being vertical (in that case the first cubes above it and below it have to be pure disordered);
- type $v$: $b_i$ is a pack of vertical facets.

Note that because of the convention we took for vertical plaquettes, there are no other cases. Moreover, from bottom to top we have the following rules:

- there exists at least one signed blob, and the blob-signs are alternating;
- the first and the last signs are $-$,
- the first signed blob after a $v$-blob is of the type $h+$.

**Remark 12** *If $B_c$ is made of exactly one horizontal plaquette, there is only one blob in $c$, and it is of type $h-$. This blob is called trivial.*

### 3.2.1 Defining Defects

Let us now consider the set $F(\sigma)$ of all frustrated cubes, attached to $B(\sigma)$. We will denote by $F_c(\sigma)$ the intersection $F(\sigma) \cap c$. Let $C_N(\sigma)$ be the set of all columns $c$, such that $B_c$ contains at least two plaquettes. For $c \in C_N(\sigma)$ let $F_i \subset F_c(\sigma)$, $i = 1, ..., r'$ be connected components of $F_c(\sigma)$. These segments of frustrated cubes will be called *defects* of $\sigma$. Now, every blob $b_j$ is contained in some defect $F_i$, but since some $F_i$-s can contain several blobs, we have $r' \leq r$. The set of all defects of $\sigma$ is denoted by $\pi(\sigma)$, while $\pi_c(\sigma) \subset \pi(\sigma)$, $c \in C_N(\sigma)$ will be those belonging to the column $c$.

Our immediate goal will be the proof of the following

**Proposition 13** *Let $D \subset \Lambda_{N,L}$ be any collection of cubes. Then*

$$\mu_{N,L}^{\beta,q}(\sigma : D \subset \pi(\sigma)) \leq a^{|D|-|\Pi(D)|}, \tag{9}$$

*where $a = a(q)$ goes to $0$ when $q \to \infty$.*

The Peierls estimate evidently follows from this.

The rough idea of proving the Proposition 13 is the following. We will try to show that the cost of having a defect with $k$ frustrated cubes is of the

order of $c^k$, $c < 1$. This is indeed true, and we will show that for all defects with $k \geq 2$ the price behaves as $c^{k-1}$. However, for some defects with $k = 1$ there is no price to pay at all, due to our choice of boundary conditions, which force the interface - and hence the defects - into the system. We will show then that if there are several such problematic defects – i.e. defects with $k = 1$ – then one can pair them, and extract the cost contribution of the order of $c$ for every pair. This will be enough for our purposes.

NOTE. The reader who would like to understand first the easy part of the proof – the one dealing with non-problematic defects – can go after the Definition 14 below straight to the Section 3.3.1.

To implement the above strategy we need to impose some more structure on the defects. First of all, we define their signs. Namely, each defect $F$ contains several blobs. Let us add all the signs of all the blobs in $F$. The resulting sign will be called the sign of $F$, $\mathrm{sgn}(F)$. It takes values $+, -$ or $0$. Since in the string of blobs in $c$ the signs are alternating, the sign of $F$ is well defined.

We will also need some information about the vicinity of the defects. So we will spatially extend the defects, fixing to a certain extent the configuration at their ends. Then, of course, we will have to perform the summation over all extensions. In the process of extension some defects might coagulate into a single bigger defect, in which case we always will treat the result as a single defect.

### 3.2.2 Extending Defects

Here we will describe the process of extending the defects. The extension will depend on $\sigma$, of course.

On the first step we extend each defect $F_j \subset c$ to a longer segment of cubes $\phi_1(F_j)$, $F_j \subset \phi_1(F_j) \subset c$, which is a minimal segment containing $F_j$, which contains, apart from $F_j$, only frustrated cubes, except two end-cubes, which are pure cubes. (The added cubes need not touch the interface.) In the case that the defect $F$ is attached to the boundary of $\Lambda_{N,L}$, the extended defect has at most one pure end-cube. Evidently, the operation $\phi_1$ is well-defined. It can happen that some resulting segments $\phi_1(F_j)$ and $\phi_1(F_{j'})$ have an elementary pure cube in common. In that case we merge them into a single defect: we will consider the connected components of the family $\{\phi_1(F_i)\}$, and by a slight abuse of terminology we still call the resulting segments defects (or extended defects). The sign of the merger is defined to be the

sum of the constituents. Now any two defects have no cubes in common (though they can share a facet). From now on we will deal exclusively with extended defects, so in what follows we will omit the symbol $\phi_1$ and will write just $F$ for the extended defects.

We also *fix the nature of every bond in the defect*, i.e. whether the bond is ordered or disordered.

**Definition 14 (Problematic defects)** *Among the defects we single out those with the property that every bond not belonging to the two end-cubes is disordered. (Note that at least one of these end-cubes has then to be ordered.) If this defect is signed, it was built from a blob consisting of just one horizontal plaquette; if it is not signed, it was built from the coagulation of two consecutive signed blobs, both consisting of just one horizontal plaquette. In both cases these defects will be called* problematic. *If both end-cubes of a problematic defect are ordered, the defect consists of 5 cubes, 3 of which are pure; if only one end-cube is ordered, the defect consists of 3 cubes, 2 of which are pure.*

*Other defects, which will be called exceptional problematic defects, appear among defects attached to the bottom (disordered) boundary. Such a defect is called e-problematic, if it has the following three properties:*

*1. It consists from one or two frustrated cubes, followed by one ordered cube at the top of the defect,*

*2. The bottom cube has at least 3 vertical disordered bonds,*

*3. The corresponding blob consists of exactly one plaquette, which is the horizontal plaquette at the bottom of the box $\Lambda_{N,L}$.*

*In particular, any e-problematic defect has sign $(-)$.*

*All other defects will be called* non-problematic.

See Figures 1,2 for (two-dimensional!) sketches of non-problematic and problematic defects.

Thus we have assigned to every configuration $\hat{\sigma}$ with $B(\hat{\sigma}) = B(\sigma) = B$ and to every column $c \in C_N(\sigma)$ the extension $\hat{\pi}_c$ of the initial set $\pi_c(\sigma)$, including into the extension the order-disorder specification of every bond of $\hat{\pi}_c$. The set of all possible extensions $\hat{\pi}$ of $\pi$ will be denoted by $\mathcal{E}(\pi)$. Evidently, we have the partition

$$\{\sigma : B(\sigma) = B, \pi(\sigma) = \pi\} = \cup_{\hat{\pi} \in \mathcal{E}(\pi)} \{\sigma : B(\sigma) = B, \hat{\pi}(\sigma) = \hat{\pi}\},$$

so

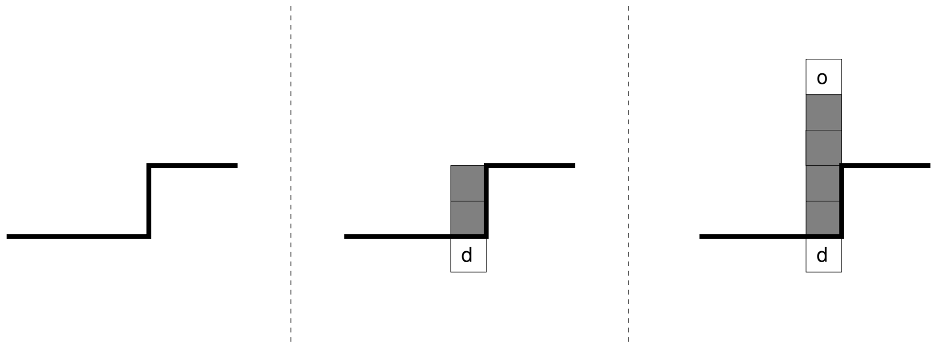


Figure 1: A piece of interface, which generates a (non-problematic) defect; frustrated cubes are shaded. The third picture shows one possible outcome of the extension of the defect.

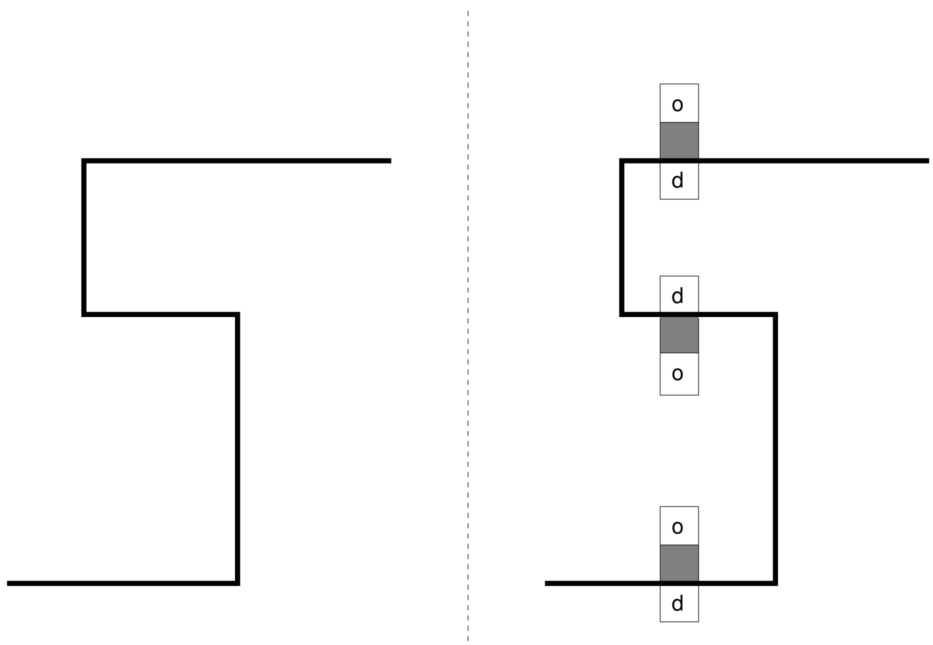


Figure 2: A piece of interface, which generates three problematic defects.

$$\mu_{N,L}^{\beta,q}\left(\sigma : \pi\left(\sigma\right)=\pi\right)=\sum_{\hat{\pi}\in\mathcal{E}\left(\pi\right)}\mu_{N,L}^{\beta,q}\left(\sigma : \hat{\pi}\left(\sigma\right)=\hat{\pi}\right).$$

We will also use the notation $\sigma\in\hat{\pi}$, in the obvious sense. A straightforward combinatorial counting of the possible extensions of a given defect shows that to prove (9) it is enough to show that

$$\mu_{N,L}^{\beta,q}\left(\sigma : \hat{\pi}\left(\sigma\right)=\hat{\pi}\right)\leq a^{\|\hat{\pi}\|-|\Pi(\hat{\pi})|} \tag{10}$$

(with some smaller $a$), where $\|\hat{\pi}\|$ is the number of frustrated cubes in $\hat{\pi}$, and $|\Pi\left(\hat{\pi}\right)|$ is the number of plaquettes in the projection $\Pi\left(\hat{\pi}\right)$.

#### 3.2.3 Fixing the boundary conditions for defects

The last phase of fixing the environment of the defect consists in fixing the type of the configuration on ordered plaquettes $P$ at the boundaries of the defect. If the plaquette $P=(x,y,z,w)$ is fully ordered, with $\sigma(x)-\sigma(y)=a$, $\sigma(y)-\sigma(z)=b$, $\sigma(z)-\sigma(w)=c$, and $\sigma(w)-\sigma(x)=d$, we say that $\sigma$ is of $(a,b,c,d)$-type on $P$; we notice that since $a,b,c,d\in\{-1,0,1\}$, there are at most $3^4=81$ possible ordered types for $\sigma$ on $P$. We denote by $\mathcal{T}$ the set of all possible types.

Each defect $F$ is delimited by two horizontal plaquettes: the top one, $F^t$, and the bottom one, $F^b$ ; we define $\partial F=F^t\cup F^b$. Each of these plaquettes can be either fully ordered or fully disordered; we denote by $\partial^o F\subset\partial F$ the ordered plaquettes of $\partial F$ (the subset $\partial^o F$ depends on $\hat{\pi}$).

For every collection $\hat{\pi}$ of extended defects, $\hat{\pi}\in\mathcal{E}\left(\pi\right)$, we define $\partial\hat{\pi}=\cup_{F\in\hat{\pi}}\partial F$ and $\partial^o\hat{\pi}=\cup_{F\in\hat{\pi}}\partial^o F$. We refine the partition $\mathcal{E}\left(\pi\right)$ by specifying the types the configuration $\sigma$ has on every plaquette from the set $\partial^o\hat{\pi}$ : if $\tau\in\mathcal{T}^{\partial^o\hat{\pi}}$, we define

$$\hat{\pi}_\tau=\left\{\sigma\in\hat{\pi} : \forall P\subset\partial^o\hat{\pi},\, \sigma \text{ is of type } \tau(P) \text{ on } P\right\},$$

so

$$\hat{\pi}=\bigcup_{\tau\in\mathcal{T}^{\partial^o\hat{\pi}}}\hat{\pi}_\tau.$$

We notice that for any column $c\in C_N\left(\sigma\right)$, containing non-trivial blob, we have for the corresponding defect, that the number of plaquettes $|\partial^o\hat{\pi}_c|\leq 3\left(\|\hat{\pi}_c\|-1\right)$ (with equality iff $\hat{\pi}_c$ consists of two problematic defects – the

first one with order-disorder b.c., the second with order-order b.c. and with one frustrated cube each). In particular,

$$\mu_{N,L}^{\beta,q}\left(\sigma:\hat{\pi}\left(\sigma\right)=\hat{\pi}\right)=\sum_{\tau\in\mathcal{T}^{\partial^{o}\hat{\pi}}}\mu_{N,L}^{\beta,q}\left(\hat{\pi}_{\tau}\right)\leq\left(81\right)^{3\left(\|\hat{\pi}\|-\left|\Pi\left(\hat{\pi}\right)\right|\right)}\sup_{\tau\in\mathcal{T}^{\partial^{o}\hat{\pi}}}\mu_{N,L}^{\beta,q}\left(\hat{\pi}_{\tau}\right). \tag{11}$$

(The estimate (11) is helpful in the discrete case, since the reflected event $\left(\hat{\pi}_{\tau}\right)_{c}^{N}$ (see below) has a relatively simple structure. This is not so in the continuous symmetry case.)

In the following $\hat{\pi}\in\mathcal{E}\left(\pi\right)$ and $\tau\in\mathcal{T}^{\partial^{o}\hat{\pi}}$ will be fixed, and we will estimate from above the $\mu_{N,L}^{\beta,q}$-probability of the event $\left\{\sigma\in\hat{\pi}_{\tau}\right\}$. We have

$$\left\{\sigma\in\hat{\pi}_{\tau}\right\}=\cap_{c\in C_{N}\left(\sigma\right)}\left\{\sigma\in\left(\hat{\pi}_{\tau}\right)_{c}\right\}, \tag{12}$$

where the event $\left(\hat{\pi}_{\tau}\right)_{c}$ consists of configurations $\sigma$ which in the column $c$ have their pattern of extended defects equal to $\hat{\pi}_{c}$, while their restriction to the plaquettes $\partial^{o}\hat{\pi}\cap c$ have types defined by $\tau_{c}\equiv\tau\Big|_{\partial^{o}\hat{\pi}\cap c}$.

The application of the chess-board estimate (see [FILS], relation (4.4)) reduces the problem of getting the upper bound for the probability $\mu_{N,L}^{\beta,q}\left\{\sigma\in\hat{\pi}_{\tau}\right\}$ to that for all probabilities $\mu_{N,L}^{\beta,q}\left\{\sigma\in\left(\hat{\pi}_{\tau}\right)_{c}^{N}\right\}$, $c\in C_{N}\left(\sigma\right)$, where the event $\left(\hat{\pi}_{\tau}\right)_{c}^{N}$ is the result of applying multiple reflections to $\left(\hat{\pi}_{\tau}\right)_{c}$. (The reflected event $\left(\hat{\pi}_{\tau}\right)_{c}^{N}$ is described in details in the following subsection.) Namely, the chess-board estimate claims that

$$\mu_{N,L}^{\beta,q}\left\{\sigma\in\hat{\pi}_{\tau}\right\}\leq\prod_{c}\left[\mu_{N,L}^{\beta,q}\left\{\sigma\in\left(\hat{\pi}_{\tau}\right)_{c}^{N}\right\}\right]^{\frac{1}{N^{2}}}. \tag{13}$$

We will prove that uniformly in $\tau$

$$\mu_{N,L}^{\beta,q}\left(\left(\hat{\pi}_{\tau}\right)_{c}^{N}\right)\leq a^{N^{2}\left(\|\hat{\pi}_{c}\|-1\right)}, \tag{14}$$

provided that "the interface $B$ is not regular in the column $c$"; that means that for any $\sigma\in\left(\hat{\pi}_{\tau}\right)_{c}$ the collection of blobs $\mathfrak{B}(B_{c})=(b_{1},\ldots,b_{r})$ of the interface $B\left(\sigma\right)$ in the column $c$ for the collection $B\left(\sigma\right)$ is not just one trivial blob. (We do not care for the situation with the trivial blob, since it does not contribute to (9) anyway.) We will call such patterns non-trivial. Then (14), (13) and (11) imply the relation (10).

### 3.2.4 Description of the reflected event $(\hat{\pi}_\tau)_c^N$

The column $c$ is now fixed. The event $(\hat{\pi}_\tau)_c$ consists of collection of (extended) defects $F_1, F_2, \ldots, F_s$ in the column $c$, each of these equipped with a boundary condition $\tau_i \in \mathcal{T}^{\partial^o F_i}$. Let the slab $\Lambda_i = \{(x,y,z);\ 0 \leq x, y \leq N,\ a_i \leq z \leq b_i\}$ be the smallest one containing the defect $F_i$. The event $\sigma \in (\hat{\pi}_\tau)_c^N$ happens if the following two conditions hold:

- in every column $c'$ the pattern of order/disorder bonds of configuration $\sigma$ agrees with $\theta_{c,c'}(F_1, F_2, \ldots, F_s)$, where $\theta_{c,c'}$ is any composition of the reflections in the lines passing through the sites of the torus, which takes $c$ to $c'$,
- on every ordered plane $z = a_i$ (resp. $z = b_i$), $i = 1, ..., s$, the configuration $\sigma$ is of "reflected" type $\left(\tau_i^b\right)^N$ (resp. $(\tau_i^t)^N$), where in column $c'$ the type $\left(\tau_i^b\right)^N$ is defined to be $\theta_{c,c'}\left(\tau_i^b\right)$ (resp. $\theta_{c,c'}(\tau_i^t)$).

We denote by $F_i^N$ the repeated reflection of the defect $F_i$, i.e. $F_i^N = \cap_{c'} \theta_{c,c'}(F_i)$. It is a pattern of order/disorder bonds in $\Lambda_i$. We put $L_i = b_i - a_i - 1$, and we define $m_i$ to be the number of frustrated cubes in $F_i$. Since every point $(x,y,z)$ with $a_i < z < b_i$ belongs to at least one frustrated cube of $F_i^N$, we have

$$L_i \leq 2m_i, \tag{15}$$

which will be of importance later. The complement $\Lambda_{N,L} \setminus (\cup_{i=1}^s \Lambda_i)$ is a collection of slabs $\Lambda_i' = \{(x,y,z);\ 0 \leq x, y \leq N,\ b_i \leq z \leq a_{i+1}\}$, $i = 0, ..., s$, with the conventions that $b_0 = 0$ and $a_{s+1} = L + 1$.

We now fix the values $\eta$ of the configuration $\sigma$ on $\partial (\hat{\pi}_\tau)_c^N$, i.e. on each plane $z = a_i$ or $z = b_i$. The set of $\eta$-s which are compatible with $(\hat{\pi}_\tau)_c^N$ is denoted by $\mathcal{B}\left((\hat{\pi}_\tau)_c^N\right)$. We choose some $\eta \in \mathcal{B}\left((\hat{\pi}_\tau)_c^N\right)$, and define

$$(\hat{\pi}_\tau)_c^N(\eta) = (\hat{\pi}_\tau)_c^N \cap \left\{ \sigma \Big|_{\partial \pi_c^N} = \eta \right\}.$$

We obviously have

$$\mu_{N,L}^{\beta,q}\left((\hat{\pi}_\tau)_c^N\right) = \sum_{\eta \in \mathcal{B}\left((\hat{\pi}_\tau)_c^N\right)} \mu_{N,L}^{\beta,q}\left((\hat{\pi}_\tau)_c^N(\eta)\right). \tag{16}$$

Uniformly in $\tau, \eta$, we will get an estimate on $\mu^{\beta,q}_{N,L}\left(\left(\hat{\pi}_\tau\right)^N_c(\eta)\right)$.

We will denote by $\eta^b_i$ (resp. $\eta^t_i$) the restriction of $\eta$ to the plane $z = a_i$ (resp. $z = b_i$). Clearly, the partition function $Z^{\beta,q}_{N,L}\left(\left(\hat{\pi}_\tau\right)^N_c(\eta)\right)$, computed over the set $\left\{\sigma \in \left(\hat{\pi}_\tau\right)^N_c(\eta)\right\}$, factors:

$$Z^{\beta,q}_{N,L}\left(\left(\hat{\pi}_\tau\right)^N_c(\eta)\right) = \prod_{i=1}^{s} Z^{\eta^b_i,\eta^t_i}_{\Lambda_i}\left(F^N_i\right) \prod_{i=0}^{s} Z^{\eta^t_i,\eta^b_{i+1}}_{\Lambda'_i}, \tag{17}$$

where the superscripts in the partition functions denote the corresponding boundary conditions for slabs (with the convention that $\eta^t_0 = \sigma_{\text{disord}}$, $\eta^b_{s+1} = \sigma_{\text{ord}}$), while the presence of arguments $F^N_i$ describe the corresponding periodic order-disorder pattern of bonds. (We note for clarity that it can happen that $b_i = a_{i+1}$ for some $i$, in which case the slab $\Lambda'_i$ degenerates to a plane, and the partition function $Z^{\eta^t_i,\eta^b_{i+1}}_{\Lambda'_i}$ is taken over the empty set; we put it to be 1 by definition.)

Our goal is now to prove that

$$\prod_{i=1}^{s} Z^{\eta_i}_{\Lambda_i}\left(F^N_i\right) \leq a^{N^2\left[\left(\sum_{i=1}^{s} m_i\right)-1\right]} \prod_{i=1}^{s} Z^{\eta_i}_{\Lambda_i}, \tag{18}$$

where $m_i$ is the number of frustrated cubes in $F_i$, and we use the shorthand notation $Z^{\eta_i}_{\Lambda_i}\left(F^N_i\right) \equiv Z^{\eta^b_i,\eta^t_i}_{\Lambda_i}\left(F^N_i\right)$, $Z^{\eta_i}_{\Lambda_i} \equiv Z^{\eta^b_i,\eta^t_i}_{\Lambda_i}$. Since, obviously,

$$\sum_{\eta \in \mathcal{B}\left(\left(\hat{\pi}_\tau\right)^N_c\right)} \frac{\prod_{i=1}^{s} Z^{\eta_i}_{\Lambda_i} \prod_{i=0}^{s} Z^{\eta^t_i,\eta^b_{i+1}}_{\Lambda'_i}}{Z^{\beta,q}_{N,L}} \leq 1,$$

the relations (18) and (16) imply (14).

We can easily deal with each non-problematic defect $F_i$, and we will show that they satisfy the estimate:

$$Z^{\eta_i}_{\Lambda_i}\left(F^N_i\right) \leq a^{m_i N^2} \cdot Z^{\eta_i}_{\Lambda_i}. \tag{19}$$

However, no reasonable estimate can be obtained for a single problematic defect. To produce the cost factor needed, we will have to treat the problematic defects in pairs, and we will produce a factor $a^{2N^2}$ for every such pair.

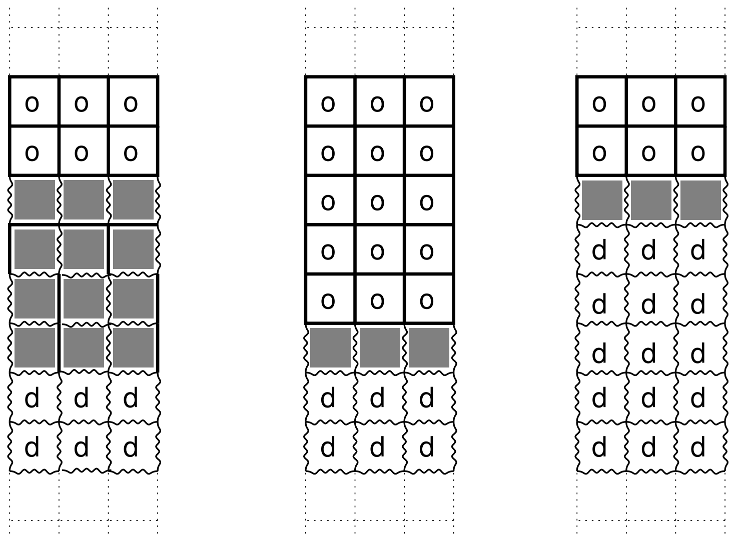


Figure 3: From left to right: a defect sheet of width four and its replacements at low and high temperatures. Ordered (disordered) bonds are represented by straight (wriggled) segments.

Let us explain the heuristics behind the above claim. Consider for example a non-problematic defect, which, in ascending order, has the following pattern of cubes (see Figure 3):

$$(..., d, d, f, f, f, f, o, o, ...)\,,$$

which means that we consider a defect sheet of width 4, sandwiched between the disordered and ordered phases. We will show in the Section 3.3.1 that the replacement of it by one of the two following narrower defect sheets:

$$(..., d, d, d, d, d, f, o, o, ...)$$

or

$$(..., d, d, f, o, o, o, o, o, ...)$$

leads to the increase of the probability. Which of the last two patterns gives the increase needed depends on the temperature; in the high temperature region the first scenario (the advance of disorder) wins over the frustration strip, while at low temperatures the second one takes over the frustration.

Moreover – and that is of crucial importance – the two temperature regions are intersecting, and at the common temperature each of the two scenarios gets a higher probability than the thick frustration sheet. Note also, that the frustration sheet can not disappear completely: in every column there should be at least one frustrated cube between the ordered and the disordered phase, which is the reason for the problematic defects to be treated separately.

In more details, our strategy will be the following: we consider all signed defects in the column $c$, which from now on will have their special notation: $G_1, G_2, ..., G_{2k-1}$. Note that we always have an odd number of them; moreover, their signs alternate, with $\mathrm{sgn}(G_1) = (-)$. Some of $G_i$-s can be problematic. The remaining neutral defects will be denoted by $H_1, H_2, \ldots, H_l$; some of them can also be problematic. We pair signed defects as follows: $(G_1, G_{2k-1}), (G_2, G_{2k-2}), \ldots$ while neutral defects are paired in the following way: $(H_1, H_2), (H_3, H_4), \ldots$ If $l$ is odd, we finally pair the remaining neutral defect $H_l$ with $G_k$; if $l$ is even, the defect $G_k$ is left unpaired. Notice that the two paired signed defects have the same sign. Note also that for a non-trivial pattern it can not happen that we have just one defect of problematic or e-problematic type.

The above pairing will be essential for us only when both defects in the pair are problematic – i.e. when we have a *problematic pair*. In that case we will treat them together via gluing construction, explained below. The pairing of the remaining defects is inessential, since each pair contains at least one non-problematic defect, so we can distribute the cost of the latter over the pair. In particular, if both are non-problematic, we will just add the two separate contributions.

### 3.2.5 Gluing process

In this section we will construct for every layered event $(\hat{\pi}_\tau)_c^N$ another layered event, $\phi_2\left((\hat{\pi}_\tau)_c^N\right)$, of a similar type. The new event will have less frustrated layers, and, what is most important, it will have no problematic pairs of defects. More precisely, we prove the following:

**Lemma 15** *For any event $(\hat{\pi}_\tau)_c^N$ with $l$ problematic pairs one can construct the event $\phi_2\left((\hat{\pi}_\tau)_c^N\right)$, such that:*

*1.*

$$\mu_{N,L}^{\beta,q}\left((\hat{\pi}_\tau)_c^N\right) \leq a^{2lN^2}\mu_{N,L}^{\beta,q}\left(\phi_2\left((\hat{\pi}_\tau)_c^N\right)\right),$$

2. *all defects of* $\phi_2\left((\hat{\pi}_\tau)_c^N\right)$ *can be paired in such a way that no pair is problematic,*

3. *the number of frustrated layers in* $\phi_2\left((\hat{\pi}_\tau)_c^N\right)$ *is* $\|\hat{\pi}_c\| - 2l$*.*

**Proof.** We proceed by induction on the number $l$ of problematic pairs, successively removing every such pair and producing instead a factor $a^{2N^2}$.

We consider first the case when the two defects paired are problematic (or e-problematic) signed defects $G_i$ and $G_{2k-i}$, with $1 \leq i \leq k-1$. We assume that the sign of $G_i$ (and therefore of $G_{2k-i}$) is minus; the plus case is even simpler, since both defects are then non-exceptional problematic defects.

We remind the reader that $G_{2k-i}$ consists of a sequence of 3 cubes: in ascending order we first meet one pure disordered cube, then one frustrated, followed by one pure ordered cube. All the bonds not in the ordered cube are disordered. $G_i$ may be of problematic or e-problematic type, when $i = 1$. In the first case it consists of $l_i = 3$ cubes. In the second case it will be convenient for us to include in the count of the cubes also the "virtual" disordered cube in the layer $\{-1 \leq z \leq 0\}$, so we put $l_1$to be 3, when the e-defect has one frustrated and one ordered cube, and we put $l_1 = 4$ when the e-problematic defect has two frustrated cubes plus one ordered on the top. Note that in any case the first frustrated cube of the defect has at least 3 vertical disordered bonds. Each of $G_j$-s comes with the boundary condition – a configuration $\eta_j \in \Omega_{\partial G_j}$.

**The first step** of the gluing process is to make a global rotation, $\Phi_1$, of the spin system in the slab $S_i = \{a_i + 2 \leq z \leq a_{2k-i}\}$, so as to make the configuration $\eta_i^t$– the configuration on the plane $\{z = b_i\}$ ,the top boundary condition of the lower defect $G_i$– to be closer to $\eta_{2k-i}^t$, the top boundary condition of the defect $G_{2k-i}$. If the defect $G_1$happens to be an e-problematic defect, then the slab $S_1 = \{1 \leq z \leq a_{2k-1}\}$ by definition.

The configurations $\eta_i^t$ and $\eta_{2k-i}^t$ are two periodic ordered configurations, defined by their restriction to any given plaquette, so we write symbolically that $\eta_i^t = (s_1, s_2, s_3, s_4)$ and $\eta_{2k-i}^t = (s_1', s_2', s_3', s_4')$, where all $s_1, ..., s_4'$ are just points of the discrete circle $\mathbb{Z}_q$. Since $\eta_i^t$ and $\eta_{2k-i}^t$ are ordered, we can choose $s \in \{s_1, s_2, s_3, s_4\}$ and $s' \in \{s_1', s_2', s_3', s_4'\}$ such that for all $j = 1, 2, 3, 4$

$$|s - s_j| \leq 1, \text{ and } \left|s' - s_j'\right| \leq 1.$$

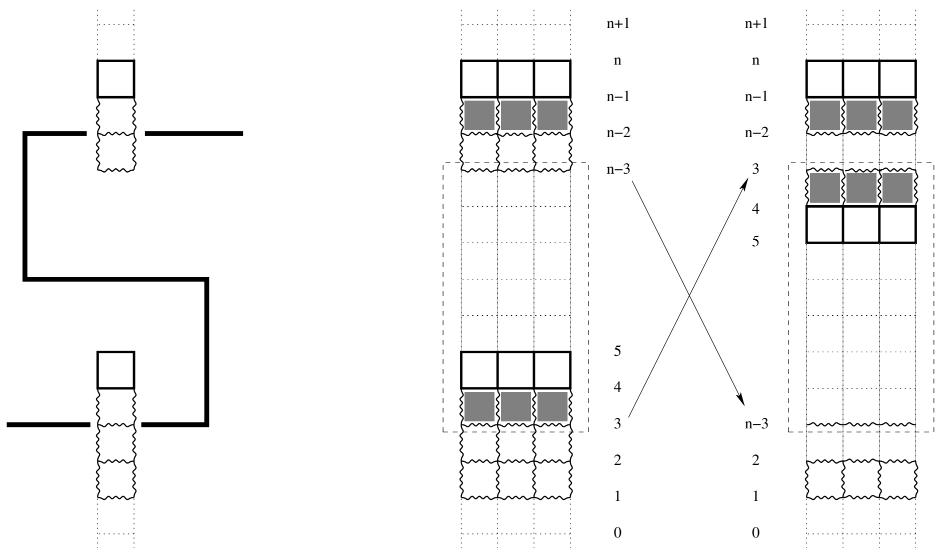


Figure 4: Second step of the gluing transformation. From left to right: two problematic defects (with minus signs), generated by a piece of interface; the corresponding reflected event (made of two defect sheets); the result of the gluing operation.

We will call the values $s, s'$ the dominant values of the boundary conditions. Now for every $\sigma \in (\hat{\pi}_\tau)_c^N$ we define $\Phi_1(\sigma)$ by

$$[\Phi_1(\sigma)](x,y,z) = \begin{cases} \sigma(x,y,z) + (s - s') & \text{if } z \in [a_i + 2, a_{2k-i}], \\ \sigma(x,y,z) & \text{otherwise.} \end{cases}$$

The transformation $\Phi_1$ is bijective.

The result thus achieved is that the configurations $\Phi_1(\eta_i^t)$ and $\Phi_1\left(\eta_{2k-i}^t\right)$ are relatively close to each other.

**The second (and the last) step** of the gluing process is to apply to the system in the slab $S_i$ the reflection $\Phi_2$ in its middle horizontal plane, thus bringing the upper part of $\Phi_1(G_i)$ in contact with $\Phi_1(G_{2k-i})$:

$$[\Phi_2(\sigma)](x,y,z) = \begin{cases} \sigma(x,y,a_i + a_{2k-i} + 2 - z) & \text{if } z \in [a_i + 2, a_{2k-i}], \\ \sigma(x,y,z) & \text{otherwise.} \end{cases}$$

See Figure 4 for a sketch of this second step. (Again, for $G_1$ being e-problematic, the reflection is done in the slab $S_1 = \{1 \le z \le a_{2k-1}\}$, with

respect to the plane $\left\{z = \frac{1+a_{2k-1}}{2} + 1\right\}$.) The composition $\Phi = \Phi_2\Phi_1$ is bijective. Note that for every configuration $\sigma \in (\hat{\pi}_\tau)_c^N$ all the bonds connecting the slab $S_i$ with its outside are disordered, except at most $\frac{N^2}{4}$ vertical bonds when $G_1$ is an e-problematic defect. Thus $\Phi$ can increase the energy of the resulting configuration by $\frac{N^2}{4}$ units, which is the possible number of ordered bonds turning into disordered ones after the rotation: $H(\sigma) - H(\Phi(\sigma)) \geq -\frac{N^2}{4}$. Therefore we get

$$\mu_{N,L}^{\beta,q}\left((\hat{\pi}_\tau)_c^N(\eta)\right) \leq e^{\frac{\beta N^2}{4}} \mu_{N,L}^{\beta,q}\left(\Phi\left[(\hat{\pi}_\tau)_c^N(\eta)\right]\right). \tag{20}$$

Let us describe the event $\Phi\left((\hat{\pi}_\tau)_c^N(\eta)\right)$. Consider the images $\tilde{F}_j^N = \Phi\left(F_j^N\right)$. If $F_j$ is between $G_i$ and $G_{2k-i}$, it is clear that $\tilde{F}_j$ has exactly the same properties as $F_j$, up to shift and reversal of pattern. Moreover, we will have $\tilde{\eta}_j^t = \Phi\left(\eta_j^b\right)$, $\tilde{\eta}_j^b = \Phi\left(\eta_j^t\right)$ as boundary conditions around $\tilde{F}_j$. If $F_j$ is before $G_i$ of after $G_{2k-i}$, we have $\tilde{F}_j = F_j$ and $\tilde{\eta}_j = \eta_j$. The pattern $\tilde{\tau} = \Phi(\tau)$ is defined in the following natural way: it coincides with $\tau$ outside the slab $S_i$, and with a reflection of $\tau$ inside this slab.

We will now focus on what happened to $G_i^N$ and $G_{2k-i}^N$ ; we denote by $\tilde{G}_{2k-i}^N$ the restriction of $\Phi\left((\hat{\pi}_\tau)_c^N\right)$ to the slab

$$\tilde{\Lambda}_{2k-i} = \{a_{2k-i} - l_i + 2 \leq z \leq b_{2k-i} = a_{2k-i} + 3\},$$

which is at most 5-cubes wide, since $l_i \leq 4$.

If $\tilde{G}_{2k-i}^N$ occurs, we have two slabs – $\{a_{2k-i} - l_i + 2 \leq z \leq a_{2k-i} - l_i + 3\}$ and $\{a_{2k-i} + 2 \leq z \leq a_{2k-i} + 3\}$ – filled with ordered bonds, and one slab – $\{a_{2k-i} + 1 \leq z \leq a_{2k-i} + 2\}$ – filled with disordered bonds; actually, the pattern of the bonds is fixed, except for the $N^2$ vertical bonds of the slab $\{a_{2k-i} \leq z \leq a_{2k-i} + 1\}$. Since the boundary conditions $\tilde{\eta}_{2k-i} = \left(\Phi\left(\eta_i^t\right), \eta_{2k-i}^t\right)$ around this defect are very close to each other, we will be able to derive the following estimate:

$$e^{\frac{\beta N^2}{4}} Z_{\tilde{\Lambda}_{2k-i}}^{\tilde{\eta}_{2k-i}}\left(\tilde{G}_{2k-i}^N(\eta)\right) \leq a^{2N^2} \cdot Z_{\tilde{\Lambda}_{2k-i}}^{\tilde{\eta}_{2k-i}}, \tag{21}$$

leading to

$$e^{\frac{\beta N^2}{4}} \mu_{N,L}^{\beta,q}\left(\Phi\left((\hat{\pi}_\tau)_c^N(\eta)\right)\right) \leq a^{2N^2} \cdot \mu_{N,L}^{\beta,q}\left((\tilde{\pi}_{\tilde{\tau}})_c^N(\tilde{\eta})\right), \tag{22}$$

where $(\tilde{\pi}_{\tilde{\tau}})_c^N(\tilde{\eta})$ is the event that for all $j$ such that $F_j \notin \{G_i, G_{2k-i}\}$, $\tilde{F}_j^N$ occurs, that the type on plaquettes of $\partial^o \tilde{F}_j$ is given by $\tilde{\tau} = \Phi(\tau)$ and that at the boundaries of $\tilde{F}_j^N$ the configuration agrees with $\tilde{\eta}_j^N$.

The remaining case of a pair of problematic defects $F_j, F_k$ with one of them – say, the upper one, $F_k$ – having both end-cubes ordered, is even simpler. Namely, it is enough to perform a global rotation in a suitable slab, which will make the two (ordered) boundary conditions of the defect $F_k$ close enough, as it was the case in the first step above. After that, the defect $F_k$ can be treated in precisely the same way as the defect $\tilde{G}_i^N$ of the preceding paragraph was treated. To define the rotation needed we take any slab $\{d_j \leq z \leq d_j + 1\}$ inside the defect $F_j$, which has at least 3 disordered vertical bonds. Such a slab clearly exists by definition. Then we do the rotation $\Phi$ of all the spins in the slab $\{d_j + 1 \leq z \leq a_k + 2\}$ by the angle $s^t - s^b$, where $s^t$ and $s^b$ are the dominant values of the boundary conditions $\eta_k^t$ and $\eta_k^b$ of the defect $F_k$, leaving all other spins unchanged. Since $\Phi$ does not increase the energy by more than $\frac{N^2}{4}$ units, we have reduced our case to the one already considered.

Applying the above arguments to each problematic pair, we get rid of all of them, getting a factor of $a^{2N^2}$ for each pair. We denote by $\phi_2$ the composition of the several transformations described above, which were needed through the gluing process. Then $\phi_2\left(F_j^N\right)$ will be the family of remaining reflected defects, not yet treated, with $\phi_2(\eta_c)$ being their boundary conditions. We denote by $\phi_2\left((\hat{\pi}_\tau)_c^N\right)$ the event that all these defects occur and that the configuration takes the prescribed values $\phi_2(\eta_c)$ on corresponding planes. Summarizing, the lemma follows from (20, 22), the proof of (21) being deferred to the next section. ∎

## 3.3 Estimating defects

The estimates proceed differently for problematic and non-problematic defects. We begin with the case of non-problematic ones.

### 3.3.1 Non-problematic defects: Proof of (19)

Thanks to the previous analysis, the proof of our main theorem is reduced to estimating a non-problematic defect. The analysis will be divided into three

cases, according to the nature of boundary conditions around the defect : disordered, mixed, or ordered.

In the reflected defect $F_i^N$, we denote by $K_i$ the number of chaotic sites, which are sites with 6 adjacent disordered bonds; notice that $K_i = k_i \frac{N^2}{4}$, with $k_i$ being an integer (or zero), due to the periodic structure of $F_i^N$. We denote by $D_i$ the number of inner disordered bonds of $F_i^N$, (those of the configurations $\eta_i$ are not included). Let us consider the connected components of the graph made by ordered bonds of $F_i^N$. Some of these components are vertical segments, not attached to the boundary; let $Q_i$ be their number. Again, $Q_i = q_i \frac{N^2}{4}$ with integer $q_i$. The number of other connected components of this graph is at most $2L_iN$. Indeed, every such component contains at least one full horizontal line (and there are $2L_iN$ such lines).

Note that the number of sites to which at least one ordered bond is attached is $L_iN^2 - K_i \leq 2m_iN^2$, while the number of connected components in this ordered bonds graph is at most $Q_i + 2L_iN$. We have therefore the following simple universal upper bound:

$$Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right) \leq 3^{2m_iN^2} q^{K_i+Q_i+2L_iN} e^{\beta\left((3L_i+1)N^2-D_i\right)}.$$

Indeed, let us pick a point in every connected component of the ordered bond graph. Then the factor $q^{Q_i+2L_iN}$ estimates the number of possible spin configurations $\varkappa$ on these sites, while $3^{2m_iN^2}$ is the estimate on the number of configurations on the ordered bond graph, given $\varkappa$. (If the spin value at one end of the ordered bond is fixed, then at the other end the spin can have 3 different values, see (6).) The factor $q^{K_i}$ is the number of configurations on chaotic sites. Finally, $(3L_i + 1)\, N^2 - D_i$ is the energy estimate.

We will use different lower bounds, depending on the boundary conditions and the temperature. They use some (heavy) combinatorics of the defects. We postpone the proof of the relevant combinatorial statements till the end of the paper.

**Order–disorder.** In this subsection we consider non-problematic defects with ordered boundary condition at one end of the defect and disordered boundary condition at the other. We have the bound

$$Z_{\Lambda_i}^{\eta_i} \geq (q-18)^{L_iN^2} + e^{3\beta L_iN^2};$$

here the first term estimates the partition function taken over fully disordered configurations, while the second one – the partition function taken over fully

ordered configurations. (In fact, it is enough to take just one ordered configuration.) If $e^\beta \leq q^{1/3}$, we have (omitting unimportant terms, not depending on $q$):

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_i N} e^{\beta\left((3L_i+1)N^2-D_i\right)}}{(q-18)^{L_i N^2}}$$
$$\leq 9^{m_i N^2}\left(\frac{1}{q}\right)^{\left(D_i-N^2\right)/3-K_i-Q_i-2L_i N}.$$

If $e^\beta \geq q^{1/3}$,

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_i N} e^{\beta\left((3L_i+1)N^2-D_i\right)}}{e^{3\beta L_i N^2}}$$
$$\leq 9^{m_i N^2}\left(\frac{1}{q}\right)^{\left(D_i-N^2\right)/3-K_i-Q_i-2L_i N}$$

By (26) below we can take $\alpha' > 0$ such that

$$\left(D_i - N^2\right)/3 - K_i - Q_i \geq 2\alpha' m_i N^2.$$

Since $L_i \leq 2m_i$, for all $N$ large enough and all order–disorder defects $F_i$,

$$\left(D_i - N^2\right)/3 - K_i - Q_i - 2L_i N \geq \alpha' m_i N^2.$$

Therefore, for all $\beta$ and all such defects,

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} q^{-\alpha' m_i N^2},$$

and the desired estimate is valid with $a(q) = 9q^{-\alpha'}$.

**Order–Order.** As in the order–disorder case, we have

$$Z_{\Lambda_i}^{\eta_i} \geq (q-18)^{L_i N^2} + e^{3\beta L_i N^2},$$

so we will be done by the previous analysis, if the estimate

$$\left(D_i - N^2\right)/3 - K_i - Q_i \geq 2\alpha' m_i N^2$$

still holds for the order-order case. This is indeed so, see again (26). Therefore for all $\beta > 0$

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} q^{-\alpha' m_i N^2}.$$

**Bulk Disorder–Disorder.** We have

$$Z_{\Lambda_i}^{\eta_i} \geq (q-18)^{L_i N^2} + e^{\beta(3L_i-1)N^2}.$$

If $e^{\beta} \leq q^{L_i/(3L_i-1)}$, we have (omitting unimportant terms, not depending on $q$) :

$$\begin{aligned}
\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} &\leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_iN} e^{\beta\left((3L_i+1)N^2-D_i\right)}}{(q-18)^{L_iN^2}} \\
&\leq 9^{m_i N^2} \left(\frac{1}{q}\right)^{\frac{L_i}{3L_i-1}\left[D_i-2N^2-\frac{3L_i-1}{L_i}(K_i+Q_i)\right]-2L_iN}.
\end{aligned}$$

If $e^{\beta} \geq q^{L_i/(3L_i-1)}$,

$$\begin{aligned}
\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} &\leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_iN} e^{\beta\left((3L_i+1)N^2-D_i\right)}}{e^{\beta(3L_i-1)N^2}} \\
&\leq 9^{m_i N^2} \left(\frac{1}{q}\right)^{\frac{L_i}{3L_i-1}\left[D_i-2N^2-\frac{3L_i-1}{L_i}(K_i+Q_i)\right]-2L_iN}.
\end{aligned}$$

By (27) we can take $\alpha' > 0$ such that

$$D_i - 2N^2 - \frac{3L_i-1}{L_i}(K_i+Q_i) \geq 6\alpha' m_i N^2.$$

Since $L_i \leq 2m_i$, for all $N$ large enough and all disorder–disorder defects $F_i$,

$$\frac{L_i}{3L_i-1}\left[D_i - 2N^2 - \frac{3L_i-1}{L_i}(K_i+Q_i)\right] - 2L_iN \geq \alpha' m_i N^2.$$

Therefore, for all $\beta$ and all such defects,

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} q^{-\alpha' m_i N^2},$$

and the desired estimate is valid with $a(q) = 9q^{-\alpha'}$.

**Boundary Disorder–Disorder.** We have

$$Z_{\Lambda_i}^{\eta_i} \geq (q-18)^{L_i N^2} + e^{\beta\left(3L_i - \frac{3}{4}\right)N^2}.$$

(We have $\frac{3}{4}$ in the energy estimate $\left(3L_i - \frac{3}{4}\right) N^2$ due to the fact that at least one quarter of the boundary bonds will be ordered.)

If $e^\beta \leq q^{L_i/\left(3L_i - \frac{3}{4}\right)}$,

$$\begin{aligned}
\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} &\leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_i N} e^{\beta\left((3L_i+1)N^2 - D_i\right)}}{(q-18)^{L_i N^2}} \\
&\leq 9^{m_i N^2} \left(\frac{1}{q}\right)^{\frac{L_i}{6\left(L_i - \frac{1}{4}\right)}\left[2D_i - \frac{7}{2}N^2 - \frac{6\left(L_i-\frac{1}{4}\right)}{L_i}(K_i+Q_i)\right] - 2L_i N}.
\end{aligned}$$

If $e^\beta \geq q^{L_i/\left(3L_i - \frac{3}{4}\right)}$,

$$\begin{aligned}
\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} &\leq 9^{m_i N^2} \frac{q^{K_i+Q_i+2L_i N} e^{\beta\left((3L_i+1)N^2 - D_i\right)}}{e^{\beta\left(3L_i - \frac{3}{4}\right)N^2}} \\
&\leq 9^{m_i N^2} \left(\frac{1}{q}\right)^{\frac{L_i}{6\left(L_i - \frac{1}{4}\right)}\left[2D_i - \frac{7}{2}N^2 - \frac{6\left(L_i-\frac{1}{4}\right)}{L_i}(K_i+Q_i)\right] - 2L_i N}.
\end{aligned}$$

Below in (36) we will show that for some $\alpha' > 0$

$$2D_i - \frac{7N^2}{2} - \frac{6\left(L_i - \frac{1}{4}\right)}{L_i}(K_i+Q_i) \geq 12\alpha' m_i N^2.$$

Since $L_i \leq 2m_i$, for all $N$ large enough and all disorder-disorder boundary defects $F_i$

$$\frac{L_i}{6\left(L_i - \frac{1}{4}\right)}\left[2D_i - \frac{7}{2}N^2 - \frac{6\left(L_i - \frac{1}{4}\right)}{L_i}(K_i+Q_i)\right] - 2L_i N \geq \alpha' m_i N^2.$$

Therefore for all $\beta$ and all such defects

$$\frac{Z_{\Lambda_i}^{\eta_i}\left(F_i^N\right)}{Z_{\Lambda_i}^{\eta_i}} \leq 9^{m_i N^2} q^{-\alpha' m_i N^2},$$

and the desired estimate is valid with $a(q) = 9q^{-\alpha'}$. ■

### 3.3.2 Glued pair of problematic defects: Proof of $(21, 22)$

We will analyze the defect $\tilde{G}_i^N$, generated by the gluing process, and will prove the estimates $(21, 22)$. The defect $\tilde{G}_i^N$ is at most 5-cubes wide, both end-layers are ordered, and all vertical bonds attached to the top cube are disordered; we notice that some vertical bonds in the third layer from the top may be ordered, possibly in a non-periodic way. We fix the pattern $V$ of these extra vertical ordered bonds, $\tilde{G}_i^N(V)$ denoting the restriction of $\tilde{G}_i^N$ to configurations agreeing with the pattern $V$. We will now estimate the partition function $Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}\left(\tilde{G}_i^N(V)\right)$ in its slab $\tilde{\Lambda}_i = \left\{\tilde{a}_i \le z \le \tilde{b}_i\right\}$, and write $\tilde{L}_i = \tilde{b}_i - \tilde{a}_i - 1 \le 4$.

The number of configurations in the slab $\tilde{\Lambda}_i$, such that the event $\tilde{G}_i^N(V)$ occurs, is bounded from above by $3^{\tilde{L}_i N^2} q^{K+Q+2\tilde{L}_i N}$, where $K, Q$ depend on $V$; every such configuration $\sigma^{(i)}$ has energy $H^{(i)}(\sigma^{(i)}) = D - \left(3\tilde{L}_i + 1\right) N^2$, where $D$ also depends on $V$. Combining this we get:

$$Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}\left(\tilde{G}_i^N(V)\right) \le 3^{\tilde{L}_i N^2} q^{K+Q+2\tilde{L}_i N} e^{\beta\left((3\tilde{L}_i+1)N^2 - D\right)}.$$

Now we need a lower bound on $Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}$. We will use one consisting of two contributions: the first is obtained by summing over high temperature configurations, while the second – by summing over low temperature ones.

For high temperatures, we just integrate over configurations with zero energy, the set of such configurations containing at least $(q-18)^{\tilde{L}_i N^2}$ configurations.

For low temperatures, we simply take one single configuration with minimal energy under given boundary conditions. Let us check that this minimum equals to $-(3\tilde{L}_i + 1)N^2$. Indeed, since the (periodic) configurations $(\tilde{\eta}_i)^t$ on $\left\{z = \tilde{b}_i\right\}$ and $(\tilde{\eta}_i)^b$ on $\{z = \tilde{a}_i\}$ have by construction the common dominant value, $s$, the constant configuration $\sigma_s \equiv s$ in $\left\{\tilde{a}_i + 1 \le z \le \tilde{b}_i - 1\right\}$ – the interior of $\tilde{\Lambda}_i$, taken with boundary conditions $\tilde{\eta}_i$, has all bonds in $\tilde{\Lambda}_i$ ordered.

Gathering all this we have:

$$Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i} \ge (q-18)^{\tilde{L}_i N^2} + e^{(3\tilde{L}_i+1)N^2\beta}.$$

If $e^{\beta} \leq q^{\tilde{L}_i/(3\tilde{L}_i+1)}$, we use $Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i} \geq (q-18)^{\tilde{L}_i N^2}$ to get

$$e^{\frac{1}{4}\beta N^2} \cdot \frac{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}\left(\tilde{G}_i^N(V)\right)}{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}} \leq \left(\frac{3q}{q-18}\right)^{\tilde{L}_i N^2} \left(q^{-\frac{\tilde{L}_i}{3\tilde{L}_i+1}}\right)^{D-\frac{1}{4}N^2-\frac{3\tilde{L}_i+1}{\tilde{L}_i}(K+Q+2\tilde{L}_i N)} \tag{23}$$

If $e^{\beta} \geq q^{\tilde{L}_i/(3\tilde{L}_i+1)}$, we use $Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i} \geq e^{(3\tilde{L}_i+1)\beta N^2}$ to get

$$e^{\frac{1}{4}\beta N^2} \cdot \frac{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}\left(\tilde{G}_i^N(V)\right)}{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}} \leq 3^{\tilde{L}_i N^2} \left(q^{-\frac{\tilde{L}_i}{3\tilde{L}_i+1}}\right)^{D-\frac{1}{4}N^2-\frac{3\tilde{L}_i+1}{\tilde{L}_i}(K+Q+2\tilde{L}_i N)}. \tag{24}$$

We will use the following

**Lemma 16** *For any pattern $V$ of ordered bonds in the third layer from the top, and all $N$ large enough*

$$D - \frac{3\tilde{L}_i+1}{\tilde{L}_i}(K+Q+2\tilde{L}_i N) \geq \left(\frac{1}{4}+\frac{1}{5}\right) N^2. \tag{25}$$

**Proof.** We recall that the ordered cubes at end-points of the defect are always disconnected in the ordered graph corresponding to $V$, because of the vertical disordered bonds in the second layer from the top; using Lemma 19 below we get $D - 3(K+Q) \geq N^2$. Moreover, it is clear that $K+Q \leq (\tilde{L}_i - 2)N^2$ and thus $D - \frac{3\tilde{L}_i+1}{\tilde{L}_i}(K+Q) \geq N^2 - \frac{(\tilde{L}_i-2)}{\tilde{L}_i}N^2 = \frac{2}{\tilde{L}_i}N^2$. Our lemma now follows from $\tilde{L}_i \leq 4$ since $\frac{2}{\tilde{L}_i}N^2 \geq \frac{1}{2}N^2 \geq \frac{1}{4}N^2 + \frac{1}{5}N^2 + 2\tilde{L}_i N$, provided $N$ is large enough. ■

Since there are $2^{N^2}$ possible patterns $V$, Lemma 16 together with $(23, 24)$ give for all $\beta$:

$$e^{\frac{1}{4}\beta N^2} \cdot \frac{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}\left(\tilde{G}_i^N\right)}{Z_{\tilde{\Lambda}_i}^{\tilde{\eta}_i}} \leq 2^{N^2} \left(\left(\frac{3q}{q-18}\right)^{\tilde{L}_i} q^{-\frac{1}{5}\cdot\frac{\tilde{L}_i}{3\tilde{L}_i+1}}\right)^{N^2} \leq a^{2N^2},$$

with $a(q) = \sqrt{2\left(\frac{3q}{q-18}\right)^4 q^{-3/50}}$, since $\tilde{L}_i \leq 4$ and also $\frac{\tilde{L}_i}{3\tilde{L}_i+1} \geq \frac{3}{10}$. ■

# 4 Combinatorial estimates for bulk defects

We prove here the needed combinatorial estimates on *non-problematic* defects, restricting the proof to defects in the bulk of the system (i.e. when the defect is not stuck to the bottom boundary), and divide this proof into three parts according to the nature of the boundary conditions around the defect. We introduce the number $d$, which equals the number of disordered cubes at the ends of our defect, i.e.

$$d = \begin{cases} 0 & \text{for the order-order bc,} \\ 1 & \text{for the order-disorder bc,} \\ 2 & \text{for the disorder-disorder bc.} \end{cases}$$

The case of boundary defects is more involved and is deferred to the next section.

For $d = 0$ or $d = 1$ non-problematic bulk defect $F$ with $m \geq 1$ frustrated cubes, and its reflection $F^N$ we will prove the relation

$$2D - 6K - 6Q \geq 2N^2 + \alpha m N^2 \tag{26}$$

for some universal $\alpha > 0$, where $D$, $K$ and $Q$ are the characteristics of $F^N$, introduced above. For $d = 2$ non-problematic bulk defect $F$ with $m \geq 1$ frustrated cubes we will prove

$$2D - 2\frac{(3L-1)}{L}(K + Q) \geq 4N^2 + \alpha m N^2. \tag{27}$$

We introduce the set $\mathrm{K}$ of chaotic sites and the set $\mathrm{D}$ of disordered bonds in $F$, $|\mathrm{K}| = K$, $|\mathrm{D}| = D$, and we rewrite $6K$ as a double sum

$$6K = \sum_{x \in \mathrm{K}} \sum_{e : x \in e} \mathbb{I}_{e \in \mathrm{D}},$$

to get

$$2D = 6K + dN^2 + \left|\partial^1 O\right| + 2\left|\partial^2 O\right|,$$

where $O$ is the graph of ordered bonds in $F^N$, and $\partial^n O$ denotes the set of disordered bonds with $n$ vertices belonging to $O$, $n = 1, 2$; the term $dN^2$ comes from the $dN^2$ vertical disordered bonds in the boundary chaotic cubes (this is precisely where we use the fact that the defect is in the bulk). We rewrite it as

$$2D = 6K + dN^2 + \sum_j |\partial X_j| + \sum_j \left|\partial^2 X_j\right|, \tag{28}$$

where $X_j$-s are the connected components of the ordered-bond graph of $F^N$, $\partial X_j$ is the set of disordered bonds touching $X_j$, and $\partial^2 X_j$ the set of disordered bonds with both vertex in $X_j$. When the number $m$ of frustrated cubes in the defect is small, we will use for the derivation of (26) the above relation (28) directly. For large $m$-s we will utilize its corollary, which we will derive now.

**Lemma 17** *The relation* (28) *implies that*

$$2D - 6K - 6Q \geq dN^2 + \frac{1}{2}mN^2 + 2Q. \tag{29}$$

**Proof.** If $X_j$ is a vertical segment, not touching the boundary, we have $|\partial X_j| = n_j + 6$, where $n_j$ is the number of frustrated cubes sharing a bond with $X_j$; also, $\partial^2 X_j = 0$. Let us denote the set of these $j$-s by $J$. For other components we use the estimate:

$$|\partial X_j| + \left|\partial^2 X_j\right| \geq \frac{1}{2} n_j. \tag{30}$$

To see it to hold, we first note that

$$|\partial X_j| + \left|\partial^2 X_j\right| \geq \frac{1}{4} \sum_c \left( |\partial X_j \cap c| + \left|\partial^2 X_j \cap c\right| \right), \tag{31}$$

where the summation goes over all cubes $c$, contributing to $n_j$; we have the factor $\frac{1}{4}$ due to the fact that every bond belongs to at most 4 cubes. We claim now that for every cube $c$ we have $|\partial X_j \cap c| + |\partial^2 X_j \cap c| \geq 2$. Indeed, either $c$ has at least two bonds from $\partial X_j$, or just one such bond, $f$. In the latter case, all other (eleven) bonds of $c$ belong to $X_j$, and therefore $f$ belongs not only to $\partial X_j$, but also to $\partial^2 X_j$. That proves (30) . Gathering all this leads to:

$$\begin{aligned} 2D - 6K &\geq dN^2 + 6Q + \sum_{j \in J} n_j + \frac{1}{2} \sum_{j \notin J} n_j \\ &\geq dN^2 + 8Q + \frac{1}{2} \sum_j n_j, \end{aligned}$$

since for every $j \in J$ we have $\frac{1}{2} n_j \geq 2$. Finally, every frustrated cube in $F^N$ contributes to at least one $n_j$, so we arrive to

$$2D - 6K - 6Q \geq dN^2 + \frac{1}{2}mN^2 + 2Q.$$

■

## 4.1 Order–disorder ($d = 1$): Proof of (26)

Here we consider a non-problematic defect with an ordered (disordered) cube at the top (bottom).

1. If $m \geq 3$, (29) gives
$$2D - 6(K+Q) - 2N^2 \geq \left(\frac{m}{2} - 1\right) N^2 \geq \frac{m}{6} N^2,$$
which is what we need.

2. Assume $m \leq 2$. Let us consider the ordered connected component $X_0$, containing the upper ordered cube; its boundary $\partial X_0$ has at least $N^2$ bonds, with equality if and only if $X_0$ is the result of multiple reflections of the upper ordered cube. Thus (28) shows that $2D - 6(K+Q) \geq 2N^2$, with equality if and only if $Q = 0$, $|\partial X_0| = N^2$, and the set $\{X_j\}$ consists of only one component – $X_0$. The equality therefore can occur only if the defect is problematic. Hence in the case considered $2D - 6(K+Q) \geq 2N^2 + \frac{N^2}{2}$.

■

## 4.2 Disorder–disorder ($d = 2$): Proof of (27)

Here we consider a non-problematic reflected defect $F^N$ surrounded by two chaotic layers. We want to obtain the bound (27) . In fact, for most defects the stronger statement holds:

$$2D - 6(K+Q) \geq 4N^2 + \alpha m N^2. \tag{32}$$

Indeed, the relation (29) reads

$$2D - 6K - 6Q \geq 2N^2 + \frac{1}{2} m N^2 + 2Q, \tag{33}$$

so the estimate (32) holds once $m \geq 5$. So we assume in the following that $m \leq 4$; if $K = Q = 0$, the simple fact that $D \geq 6N^2$ is enough to get (32), so we assume it is not the case. But then, it is enough to show that $2D - 6(K+Q) \geq 4N^2$; indeed, (27) will follow from $L \leq 2m = 8$ and $K + Q \geq \frac{N^2}{4}$.

Next we note the following simple

**Lemma 18** *For any bulk defect with disorder–disorder b.c. ($d = 2$), the existence of an ordered horizontal bond $e$ implies*

$$\sum_{i \notin J} |\partial X_i| \geq 2N^2.$$

**Proof.** Indeed, in its column $c$ the bond $e$ has two horizontal adjacent bonds $e', e''$. If both of them are disordered, their reflections produce $N^2$ horizontal bonds belonging to $\cup_{i \notin J} \partial X_i$, while the reflections of the bond $e$ contain $\frac{N^2}{2}$ sites, each of which has a disordered bond from $\cup \partial X_i$ above it and another one below it. If $e'$ is ordered and $e''$ is disordered, we get similarly $\frac{N^2}{2}$ horizontal bonds and $\frac{3N^2}{2}$ vertical bonds in the boundaries. If both $e'$ and $e''$ are ordered, we get $2N^2$ vertical bonds in the boundaries. ■

The previous lemma, combined with (28) , reduces the analysis to the case where there is no ordered horizontal bonds. In this last case we have $2D - 6K = 2N^2 + 6Q + \sum_{j \in J} n_j$, where $n_j$ is the number of frustrated cubes sharing a bond with $X_j$. From this, we get $2D - 6(K + Q) \geq 2N^2 + mN^2 \geq 4N^2$, if $m \geq 2$. If $m = 1$, then $L = 2$, $K = 2N^2 - 2Q$, $D = 7N^2 - Q$ with $Q \geq \frac{N^2}{4}$, so that $2D - 2\frac{3L-1}{L}(K + Q) = 2D - 5(K + Q) = 4N^2 + 3Q \geq 4N^2 + \frac{3N^2}{4}$.
■

### 4.3 Order–order ($d = 0$): Proof of (26)

We consider the reflection $F^N$ of a non-problematic defect $F$ with $m$ frustrated cubes, surrounded by two ordered cubes. Since every defect by definition contains a disordered plaquette, every order-order defect has $m \geq 2$. We want to establish the relation (26) : $2D - 6(K + Q) \geq 2N^2 + \alpha m N^2$. For $m \geq 5$ it follows immediately from (29), so we assume that $m \leq 4$. In this case the relation (26) follows from the following two lemmas:

**Lemma 19** *For all defects with order–order b.c. and such that the ordered cubes at the ends of the defect are disconnected in the ordered graph,*

$$2D - 6(K + Q) \geq 2N^2,$$

*with equality if and only if it is problematic.*

**Proof.** We denote by $X_0$ and $X_1$ the two connected components corresponding to the extreme ordered cubes. Then $|\partial X_0| + |\partial X_1| \geq 2N^2$ and (28) shows the inequality, and we see that the case of equality is precisely the problematic defect. ■

**Lemma 20** *For all defects with order–order b.c. such that ordered cubes at the ends of the defect belong to the same component,*

$$2D - 6(K + Q) \geq 3N^2.$$

**Proof.** We denote by $X_0$ the component containing both ordered cubes. Our assumption means that our defect contains a vertical disordered plaquette $P$. Indeed, all the blobs defining our defect have only vertical plaquettes, since the defect does not contain disordered cubes.

Looking at the two vertical lines passing through $P$, we see that each of them is either completely ordered outside $P$, its unique disordered bond then belonging to $\partial^2 X_0$, or else it has two bonds in $\partial X_0$; therefore, the contribution of vertical bonds to $|\partial X_0| + |\partial^2 X_0|$ is at least $2\frac{N^2}{2} = N^2$.

We shall now prove that the horizontal contribution to $|\partial X_0| + |\partial^2 X_0|$ is at least $2N^2$. Let us look at the horizontal plaquette $P'$, which contains the bottom horizontal bond of $P$. Of course, this bond is disordered. If some ordered bonds of $P'$ belong to $X_0$, then $|\partial X_0 \cap P'| + |\partial^2 X_0 \cap P'| \geq 2$, as a simple counting shows. Otherwise, since there is an ordered path through the defect, there is a vertical bond in $X_0$ touching $P'$ at a vertex $x$. By assumption, the two bonds of $P'$ containing $x$ are disordered (since otherwise they would belong to $X_0$), so they both are in $\partial X_0$. The same holds for the horizontal plaquette $P'$, which shares the top horizontal bond with $P$, which proves our claim. ■

■

# 5 Combinatorial estimates for boundary defects

Now we deal with the case when the defect is stuck to the bottom of the box.

## 5.1 Order–disorder: Proof of (26)

Let us denote by $D^b$ the number of vertical disordered bonds attached to the bottom boundary of $F^N$ and replacing in (29) the term $dN^2$ by $D^b$, we have the analog of (28)

$$2D = 6K + D^b + \sum_j |\partial X_j| + \sum_j \left|\partial^2 X_j\right|, \tag{34}$$

and the analog of (29) :

$$2D - 6K - 6Q \geq D^b + \frac{1}{2}mN^2 + 2Q. \tag{35}$$

(We remark for clarity that here $Q$ is the number of ordered vertical segments, not touching both boundaries of the defect $F^N$.)

We recall the reader that we aim to prove the relation (26) for non-e-problematic defects. Note that the only $m = 1$ boundary defect with order–disorder b.c. is e-problematic. So in what follows we assume that $m \geq 2$.

If $m \geq 5$, the relation (26) follows directly from (35). For smaller $m$ we will use the following three lemmas.

**Lemma 21** *For any $m \leq 4$ boundary defect with order-disorder b.c. the strong disorder b.c. implies that $\partial X_0$ contains at least $\frac{3}{4}N^2$ vertical bonds, provided $q$ is large enough. (Here $X_0$ is the ordered component of the top ordered cube.)*

**Proof.** Since $m \leq 4$, the defect is at most 8-cubes wide, and therefore contains at most 36 sites. If $\partial X_0$ had less than 3 vertical bonds in the column, then two sites $u, v$ of the bottom plaquette would belong to $X_0$. Denoting by $M$ the size of the largest possible path in a graph with 36 sites, we would get $|\sigma_u - \sigma_v| \leq M$, therefore contradicting the strong disorder b.c. for $q$ large enough. ∎

**Lemma 22** *Consider any boundary defect with any b.c. on the top. If $D^b \leq \frac{N^2}{4}$, the strong disorder b.c. imply*

$$\sum_{i \notin J} |\partial X_i| + \left|\partial^2 X_i\right| \geq 3N^2.$$

**Proof.** We start with the case $D^b = 0$. Since the b.c. are strongly disordered, all horizontal bonds of the first layer have to be disordered as well, and each of them belong to $\partial^2 X_i$ for some $i \notin J$, so their contribution to the sum above is $4N^2$.

In the case $D^b = N^2/4$, the strong disordered b.c. implies that three or four horizontal bonds in the first layer are disordered. If we have 4 such disordered bonds, they all belong to some $\partial X_i$, and two of them actually belong to some $\partial^2 X_i$; if we have only three such bonds, they all belong to some $\partial^2 X_i$. In any case, they contribute $3N^2$ to the sum above. ∎

**Lemma 23** *For any boundary defect with order–disorder b.c. with a horizontal disordered bond at the level $z = 1$, the contribution of horizontal bonds to*

$$\sum_{i \notin J} |\partial X_i| + \left|\partial^2 X_i\right|,$$

*is at least $N^2$.*

**Proof.** If all four bonds of the horizontal plaquette $P$ at $z = 1$ are disordered, there has to be a vertical ordered bond touching the boundary (because the first cube is frustrated). It touches two horizontal bonds of $P$; all their reflections contribute $N^2$ to the sum. If the plaquette $P$ has two or three disordered bonds, at least two of them belong to the boundary of some $X_j$. Finally, if $P$ has only one disordered bond, then it belongs to $\partial^2 X_j$ for some $j$, and so contributes twice to the sum above. ■

If $D^b \leq \frac{N^2}{4}$, we can apply Lemma 22 and (34) to get (26). If $D^b = \frac{N^2}{2}$, the strong disorder b.c. prevent the horizontal plaquette at $z = 1$ from being completely ordered, so we can apply Lemma 23 together with Lemma 21, getting (26).

If $D^b \geq \frac{3N^2}{4}$ and $m \geq 3$, we apply (35) to get the relation (26).

In the remaining case $D^b \geq \frac{3N^2}{4}$ and $m = 2$ we know, that the blob corresponding to the defect had at least 2 plaquettes, because it would be e-problematic otherwise. If this extra (disordered!) plaquette is horizontal, then the second cube is pure disordered; else it is vertical. In any case the horizontal plaquette at $z = 1$ cannot be completely ordered. Thus we can apply Lemma 23 and (34) to get (26).

■

## 5.2 Disorder–disorder: Proof of (36)

Now we prove the relation

$$2D - \frac{6\left(L - \frac{1}{4}\right)}{L}(K + Q) \geq \frac{7N^2}{2} + \alpha m N^2. \tag{36}$$

In fact, for most defects we will prove the stronger statement (32) : $2D - 6(K + Q) \geq \frac{7N^2}{2} + \alpha m N^2$. We start with the identity

$$2D - 6K = D^b + N^2 + \sum_i |\partial X_i| + \sum_i \left|\partial^2 X_i\right|, \tag{37}$$

where $D^b$ is the number of vertical disordered bonds attached to the bottom boundary of $F^N$. (The term $N^2$ equals to the number of vertical disordered bonds attached to the top boundary.) From this we deduce, as above, that

$$2D - 6K - 6Q \geq D^b + N^2 + \frac{mN^2}{2} + 2Q. \tag{38}$$

The desired estimate is directly derived from this for $m \geq 6$. We now deal with the case $m \leq 5$.

The case $m = 1$ is completely explicit. We have $L = 1$, $D = 3N^2 + D^b$, $Q = 0$, $K = D^b$ and $D^b \leq \frac{3}{4}N^2$ (because the first cube is frustrated), so that $2D - 6\frac{L-1/4}{L}K = 6N^2 + 2D^b - \frac{9}{2}D^b = 6N^2 - \frac{5}{2}D^b \geq 4N^2 + \frac{1}{8}N^2$, and thus we assume $m \geq 2$.

Also, if $K = Q = 0$, the simple fact that $D \geq 3N^2$ is enough to get (32), so we assume $K + Q \geq \frac{N^2}{4}$.

**Lemma 24** *For any boundary defect with $m \geq 2$ and all horizontal bonds disordered,*

$$D^b + \sum |\partial X_j| \geq 3N^2 + 6Q.$$

**Proof.** Our assumption implies that all ordered components are vertical segments ; since the first cube must be frustrated, $D^b \leq \frac{3}{4}N^2$. For $j \notin J$, $X_j$ starts from the boundary and $\sum |\partial X_j| \geq 10Q + \sum_{j \notin J} (4|X_j| + 1) \geq 10Q + 5|J^c|$. Since $|J^c| = N^2 - D^b$, we have $D^b + \sum |\partial X_j| \geq 10Q + 5N^2 - 4D^b \geq 10Q + 2N^2$. The lemma is then proved if $Q \geq \frac{1}{4}N^2$, so we assume $Q = 0$.

We now pick one ordered bond in the second frustrated cube, which is vertical by assumption. Since $Q = 0$, the corresponding ordered component $X_j$ satisfies $|X_j| \geq 2$. After reflections, there are $\frac{N^2}{4}$ such segments, and $\frac{3N^2}{4} - D^b$ other segments. Then $D^b + \sum |\partial X_j| \geq D^b + \frac{9}{4}N^2 + 5\left(\frac{3N^2}{4} - D^b\right) = 6N^2 - 4D^b$ and the lemma follows from $D^b \leq \frac{3}{4}N^2$. ■

**Lemma 25** *For any boundary defect with disorder-disorder b.c., and for any ordered horizontal bond $e$,*

$$|\partial X_e| + \left|\partial^2 X_e\right| \geq N^2 + \frac{3}{4}N^2,$$

*where $X_e$ denotes the ordered component containing $e$.*

**Proof.** We denote by $P$ the horizontal plaquette containing $e$. If $P$ is completely ordered, the strong disorder b.c. force $\partial X_e \geq N^2 + \frac{3}{4}N^2$ (compare with Lemma 21). If $P$ is not completely ordered, either two horizontal bonds of $P$ belong to $\partial X_e$ or one of them is in $\partial^2 X_e$ ; moreover, due to the strong disorder b.c. at least 3 vertical bonds of the column belong to the boundary of $X_e$. ■

If no horizontal bond is ordered, we can combine (37) with Lemma 24 to get $2D - 6(K+Q) \geq 4N^2$.

Otherwise, we can find a bond $e$ to which we apply Lemma 25, to get $2D - 6(K+Q) \geq 2N^2 + D^b + \frac{3}{4}N^2$. If $D^b \geq \frac{1}{2}N^2$, this shows that $2D - 6(K+Q) \geq 3N^2 + \frac{1}{4}N^2$. Since $2D - 6(K+Q)$ is an integer multiple of $\frac{1}{2}N^2$, we actually have $2D - 6(K+Q) \geq \frac{7}{2}N^2$. (36) now follows from $L \leq 2m = 10$ and $K + Q \geq \frac{N^2}{4}$.

If $D^b \leq \frac{1}{4}N^2$ we apply Lemma 22 and (37) to get $2D - 6(K+Q) \geq 4N^2$. ■

# 6 Proof of the Main Theorem 8

In this section, we derive our main results from the Peierls estimate.
We start with the question of the interface uniqueness.

**Lemma 26** *For any $b > 1$ there exists a $q_0 < \infty$ such that the following holds: For any $q \geq q_0$ and any sequence $L_N \leq b^{N^2}$ the probability $\mu^{\beta,q}_{N,L_N}$ (discn) of the event that the interface $B$ is disconnected, vanishes as $N \to \infty$.*

**Proof.** Let $\mathbf{0} \in \mathbf{T}$ be the origin. Denote by $l(B)$ the quantity $\min\{z : (\mathbf{0}, z) \in B\}$ ; it is the height of the interface $B$ at the origin. Let $m(B)$ be the number of frustrated cubes having at least one plaquette in common with $B$.

Let $B$ be disconnected. Then it has at least three connected components, which are interfaces themselves. Let $B_1, B_2, B_3$ be the first three of them. Clearly,

$$\begin{aligned}&\Pr(B \text{ is disconnected})\\&= \sum_{l_1 < l_2 < l_3} \Pr(B_1, B_2, B_3 : l(B_1) = l_1, l(B_2) = l_2, l(B_3) = l_3).\end{aligned}$$

Applying the Proposition 13 we have

$$\begin{aligned}&\Pr\left(B_1,B_2,B_3:l\left(B_1\right)=l_1,l\left(B_2\right)=l_2,l\left(B_3\right)=l_3\right)\\&\leq a^{m(B_1)+m(B_2)+m(B_3)-N^2}.\end{aligned}$$

Note that for any $B$ the number $m(B)\geq N^2$, and the number of interfaces $B$ with $m(B)=m$ and with $l(B)$ fixed is at most $C^m$ for some $C$. Therefore

$$\begin{aligned}&\Pr\left(B\text{ has at least three components}\right)\\&\leq L_N^3\sum_{m\geq 3N^2}\left(Ca^{\frac{2}{3}}\right)^m=const\cdot\left(L\left(Ca^{\frac{2}{3}}\right)^{N^2}\right)^3,\end{aligned}$$

which goes to zero as $N\to\infty$ once $L_N<b^{N^2}$ with $b<\left(Ca^{\frac{2}{3}}\right)^{-1}$. ■

In what follows we will treat only connected interfaces. We will now show that typically the interface does not have a wall which winds around the torus. The reason is that such walls contain so many plaquettes that they appear very seldom, as estimates from previous sections will show. We say that a wall $\gamma$ is winding if the projection $\Pi(\gamma)$ contains a non-trivial loop of the torus. In that case $\gamma$ contains at least $N$ plaquettes, so

$$\mu_{N,L_N}^{\beta,q}\left(\text{ there is a winding wall }\right)\leq 2N^2L_N\sum_{l\geq N}C^la^{l/2},$$

which goes to zero as $N\to\infty$, once $L_N<b^N$ with $b<\left(Ca^{1/2}\right)^{-1}$.

Let $M\in\mathbb{T}_N$ be a point in the 2D torus, and $\gamma$ be a wall of some interface in the 3D box $\Lambda_{N,L}$. Denote by $\tilde{\gamma}$ the projection $\Pi(\gamma)$. We will say that $\gamma$ surrounds $M$, iff $M\in\tilde{\gamma}\cup\operatorname{Int}(\tilde{\gamma})$. Evidently, the rigidity property of the interface that we want to prove, would follow from the

**Proposition 27**

$$\mu_{N,L}^{\beta,q}\left(M\textit{ is surrounded by some wall}\right)\leq c(q),\qquad(39)$$

*with* $c(q)\to 0$ *as* $q\to\infty$.

**Remark.** The long-range claim of our main Theorem 8 also follows from the Proposition 27. Indeed, if the heights $h(M')\neq h(M'')$ or one of them is infinite, then at least one of the points $M',M''$ is surrounded by a wall.

**Proof.** From the Peierls estimate we know that the probability of the presence of an interface wall $\gamma$ satisfies

$$\mu_{N,L}^{\beta,q}(\gamma) \leq a^{w(\gamma)}.$$

However, we need evidently the estimate on the probability of the larger event $\gamma^* = \cup_\tau \gamma^\tau$, where $\gamma^\tau$ is the wall obtained from $\gamma$ by a vertical shift along the vector $(0,0,\tau)$. Since there are about $L$ values of $\tau$ for which $\gamma^\tau \subset \Lambda_{N,L}$, the estimate we have thus far is

$$\mu_{N,L}^{\beta,q}(\gamma^*) \leq La^{w(\gamma)},$$

and since $L$ is diverging with $N$, the above estimate seems to be not enough for our purposes.

Yet, we know more about our measure $\mu_{N,L}^{\beta,q}$. Namely, we know also that if $\Gamma$ is a collection $(\gamma_1, ..., \gamma_k)$ of the walls belonging to the same interface, then

$$\mu_{N,L}^{\beta,q}(\Gamma) \leq a^{w(\Gamma)},$$

with $w(\Gamma) = \sum w(\gamma_i)$. Therefore if $\Gamma^{Ext} = (\gamma_1, ..., \gamma_k)$ is a collection of *exterior* walls in some interface, and $\Gamma^{Ext*} = (\gamma_1^*, ..., \gamma_k^*)$ is the event $\cup_{\tau_1,...,\tau_k}(\gamma_1^{\tau_1}, ..., \gamma_k^{\tau_k})$ of observing the collection $(\gamma_1^{\tau_1}, ..., \gamma_k^{\tau_k})$ to be exterior walls of an interface, then necessarily $\tau_1 = ... = \tau_k$, and so

$$\mu_{N,L}^{\beta,q}\left(\Gamma^{Ext*}\right) \leq La^{w(\Gamma)}.$$

This estimate is helpful to eliminate long walls, but it is useless for dealing with collections $\Gamma$ of walls of finite total length $l$, when $L$ is large. Note however that such a collection $\Gamma$ can surround only a finite total area $\leq l^2$. Since our measure $\mu_{N,L}^{\beta,q}$ is translation invariant, the probability to see $\Gamma$ at any given location can be estimated by $\frac{l^2}{N^2}$. In what follows we will make these heuristic arguments rigorous.

We start with the following simplified model, which contains all the essential features of our problem. Let $\mathbb{T}_R$ be a $R \times R$ discreet torus, and let $\xi_t = 0, 1$ be a random field indexed by $t \in \mathbb{T}_R$. Let $\mu$ be the distribution of the field $\xi$.

**Lemma 28** *Suppose that*

- *$\mu$ is translation-invariant,*

- *there exist a value $K$ such that for all $k \geq K$*

$$\mu\left(\xi_{t_1} = \xi_{t_2} = ... = \xi_{t_k} = 1\right) \leq \alpha^k, \tag{40}$$

*for small enough $\alpha$. Then there exists a function $R(\alpha)$, such that for any $t \in \mathbb{T}_R$*

$$\mu\left(\xi_t = 1\right) \leq 3\alpha,$$

*provided $R \geq \sqrt{\frac{K}{3\alpha}} + R(\alpha)$.*

**Proof.** Let us show first that for any $k$

$$\mu\left(\xi_0 = 1, \sum_{t\in\mathbb{T}_R} \xi_t = k\right) = \frac{k}{R^2}\mu\left(\sum_{t\in\mathbb{T}_R} \xi_t = k\right). \tag{41}$$

To see this let $X \subset \mathbb{T}_R$ be any subset with

$|X| = k$, and let $\tilde{X}$ be the event that the support $\mathrm{Supp}(\xi)$ coincides with some shift $X + t$ of $X$, $t \in \mathbb{T}_R$. Note that for any $X$

$$\mu\left(\xi_0 = 1 \,\middle|\, \tilde{X}\right) = \frac{k}{R^2}.$$

This is immediate if the set $X$ is not periodic, i.e. if all the shifts $X + t$ are different subsets. In case $X$ is periodic we have to consider the sublattice $\mathcal{L}_X \subset \mathbb{T}_R$ of its periods and its fundamental parallelogram $\mathcal{P}_X \subset \mathbb{T}_R$. By the same reasoning $\mu\left(\xi_0 = 1 \,\middle|\, \tilde{X}\right) = \frac{|X\cap\mathcal{P}_X|}{|\mathcal{P}_X|}$, while evidently $\frac{|X\cap\mathcal{P}_X|}{|\mathcal{P}_X|} = \frac{k}{R^2}$. Finally, $\mu\left(\xi_0 = 1, \sum_{t\in\mathbb{T}_R} \xi_t = k\right) = \sum_{\tilde{X}} \mu\left(\xi_0 = 1, \tilde{X}\right) = \frac{k}{R^2} \sum_{\tilde{X}} \mu\left(\tilde{X}\right) = \frac{k}{R^2}\mu\left(\sum_{t\in\mathbb{T}_R} \xi_t = k\right)$.

We now prove our lemma. From (41) we know that for all $M \geq 1$,

$$\mu\left(\xi_0 = 1, 1 \leq \sum_{t\in\mathbb{T}_R} \xi_t \leq M\right) = \sum_{k=1}^{M} \frac{k}{R^2}\mu\left(\sum_{t\in\mathbb{T}_R} \xi_t = k\right) \leq \frac{M}{R^2}. \tag{42}$$

In the region $\sum_{t\in\mathbb{T}_R} \xi_t \geq M$ we would like to use the "Peierls estimate" (40).

Our choice will be $M = bR^2$ with some $b \geq 2\alpha$. Then we have

$$\mu\left(\xi_0 = 1, \sum_{t\in\mathbb{T}_R} \xi_t \geq M\right) \leq \binom{R^2}{M}\alpha^M. \tag{43}$$

Once $R$ satisfies $bR^2 > K$, we can use both (42) and (43) to conclude that

$$\mu\left(\xi_0 = 1\right) \leq b + \binom{R^2}{bR^2} \alpha^{bR^2}.$$

Finally, introducing $c = 1 - b$, we have by Stirling, that

$$\begin{aligned} \binom{R^2}{bR^2} \alpha^{bR^2} &\sim \frac{R^{2R^2}}{\sqrt{2\pi bcR^2}\left(bR^2\right)^{bR^2}\left(cR^2\right)^{cR^2}} \alpha^{bR^2} \\ &= \frac{1}{\sqrt{2\pi bcR^2}} \left(\frac{\alpha^b}{b^b c^c}\right)^{R^2}. \end{aligned}$$

A straightforward check shows that for $b = 3\alpha$ the ratio $\frac{\alpha^b}{b^b c^c} < 1$, so $\binom{R^2}{bR^2} \alpha^{bR^2} \to 0$ as $R \to \infty$, which concludes the proof. ■

Returning to the proof of Proposition 27, we take two large numbers, $R$ and $Q$, to be chosen later, and we consider the box $\Lambda_{N,L}$ with $N = QR$. Let $O$ be the origin, $O \in \mathbb{T}_N$. The probability that $O$ is surrounded by a wall with the weight $w \geq Q$ satisfies

$$\mu_{N,L}^{\beta,q}\left(O \text{ is } Q\text{-surrounded}\right) \leq La^Q \tag{44}$$

(modulo unimportant constant). So once $Q \gg \ln L$, this probability is small. We are left with the event that $O$ is surrounded by a wall with the weight $w < Q$. To estimate its probability we will use the above Lemma. Let us consider the torus sublattice $\mathbb{T}_R \subset \mathbb{T}_N$. For every point $t \in \mathbb{T}_R$ we define the random variable $\xi_t$ by

$$\xi_t = \begin{cases} 1 & \text{if } t \text{ is inside some } \tilde{\gamma} \text{ with } w\left(\gamma\right) < Q, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, the field $\{\xi_t, t \in \mathbb{T}_R\}$ is translation-invariant. Let us estimate the probability of the event

$$\xi^T = \left\{\xi_t = 1 \text{ for all } t \in T \subset \mathbb{T}_R\right\}.$$

As was explained above, our Peierls estimate gives

$$\mu_{N,L}^{\beta,q}\left(\xi^T\right) \leq L \sum_{\gamma_1,\ldots,\gamma_{|T|}} a^{w(\gamma_1)+\ldots+w\left(\gamma_{|T|}\right)},$$

where the summation goes over all collections $\{\gamma_1, ..., \gamma_{|T|}\}$ of exterior walls with base at the given level – say, $L/2$ – such that every $\gamma_i$ surrounds precisely one point of $T$. Since always $w(\gamma) \geq 4$,

$$\mu_{N,L}^{\beta,q}\left(\xi^T\right) \leq La^{4|T|}.$$

Therefore the condition (40) holds with $\alpha = a^2$ and $K = \frac{1}{2} \ln L \ln \frac{1}{a}$. Hence for

$$R \geq \frac{\sqrt{\ln L \ln \frac{1}{a}}}{a} \tag{45}$$

we have $\mu_{N,L}^{\beta,q}(\xi_O = 1) \leq 3a^2$. On the other hand, our bound $La^Q$ from (44) satisfies $La^Q < a^2$ once

$$(Q-2)\ln\frac{1}{a} > \ln L. \tag{46}$$

If we take $Q = R^2$, then both inequalities (45) and (46) will be satisfied, provided $\ln L < N^{2/3}$. So under this condition we can conclude that (39) is satisfied with $c(q) = 4a^2$.

■

# 7 Conclusions

In this work we have developed a version of the Reflection Positivity method suitable for the investigation of the rigidity property of the interfaces between coexisting phases of certain 3D systems. It is applicable to various known models, such as the Ising, Potts or FK models. However, the main advantage of the method is that it works also for models with non-trivial structure of the ground states, which can not be treated by the PS theory, one example being the clock version of the "very non-linear $\sigma$-model".

We hope to be able to extend our methods to systems with continuous symmetry.

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